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FORMALLY SELF ADJOINTNESS FOR THE DIRAC OPERATOR ON HOMOGENEOUS SPACES

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Introduction. In [5], Wolf proved that the Dirac operator is essentially self adojoint over a Riemannian spin manifold M and he used it to give explicit realization of unitary representations of Lie groups.

Let K be a Lie group and α a Lie group homomorphism of K into $SO(n)$ which factors through Spin (n). He defined the Dirac operator on spinors with values in a certain vector bundle under the assumption that the Riemannian connection on the oriented orthonormal frame bundle P over M can be reduced to some principal K-bundle over M by the homomorphism α.

The purpose of this paper is to give the Dirac operator on a homogeneous space in a more general situation using an invariant connection, and to determine connections that define the formally self adjoint Dirac operator.

Let G be a unimodular Lie group and K a compact subgroup of G. We assume G/K has an invariant spin structure. First, we define the Dirac operator D on spinors using an invariant connection on the oriented orthonormal frame bundle P over G/K. Next, we introduce an invariant connection $\nabla^{\mathcal{V}}$ to a homogeneous vector bundle $\mathcal{V}$ associated to a unitary representation of K, then we define the Dirac operator $D \underset{\nabla^{\mathcal{V}}}{\hat{\otimes}} 1$ on spinors with values in $\mathcal{V}$ according to [4]. As for a metric on spinors, we use a Lemma given by Parthasarathy in [3]. Using this metric and an invariant measure on G/K, we define a hermitian inner product on the space of spinors with values in $\mathcal{V}$. Then we determine connections that define the formally self adjoint Dirac operator with respect to this inner product. In some cases (cf. Remarks in **4**), $D \underset{\nabla^{\mathcal{V}}}{\hat{\otimes}} 1$ is always formally self adjoint if an invariant connection on $\mathcal{V}$ is a metic connection. Moreover, in the same way as Wolf [3], we see that if $D \underset{\nabla^{\mathcal{V}}}{\hat{\otimes}} 1$ is formally self adjoint, then $D \underset{\nabla^{\mathcal{V}}}{\hat{\otimes}} 1$ and $(D \underset{\nabla^{\mathcal{V}}}{\hat{\otimes}} 1)^2$ are essentially self adjoint.

1. Spin construction

Let $\mathfrak{m}$ be an n-dimensional oriented real vector space with an inner product