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ON THE SPACES OF GENERALIZED CURVATURE TENSOR FIELDS AND SECOND FUNDAMENTAL FORMS

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For a Riemannian manifold M let $\mathfrak{A}(M)$ be the vector space of all tensor fields A of type (1,1) that satisfy the following three conditions: (1) A is symmetric as an endomorphism of each tangent space $T_x(M)$, $x \in M$; (2) Codazzi's equation holds, that is, $(\nabla_X A)Y = (\nabla_Y A)(X)$ for all vector fields X and Y; (3) trace A is constant on M. It is hardly necessary to note that an isometric immersion of M into a space of constant sectional curvature as a hypersurface with constant mean curvature gives rise to such a tensor field A (namely, the second fundamental form), which furthermore satisfies the equation of Gauss. Now Y. Matsushima has shown (unpublished) that if M is a compact Riemannian manifold, then $\mathfrak{A}(M)$ is finite-dimensional. This is obtained as an application of the theory of vector bundle-valued harmonic forms (see [2] for other applications to the study of isometric immersions).

The purpose of the present paper is to prove two results (Theorems 1 and 2) of a similar nature. Theorem 1 generalizes the above result of Matsushima to the space of generalized second fundamental forms, which, geometrically, arise from isometric immersions of higher codimension. Theorem 2 shows finite-dimensionality of the space of generalized curvature tensor fields, which, as a matter of fact, implies the above result of Matsushima as we show in [3].

1. Forms with values in a Riemannian vector bundle

By a Riemannian vector bundle we shall mean a (real) vector bundle E over a Riemannian manifold M which has a fiber metric and a mtric connection ([1], Vol. I. pp. 116–7). The Riemannian metric on M and the fiber metric in E are denoted by $\langle\ ,\ \rangle$, whereas the Riemannian connection on M is denoted by ∇ and the metric connection in E by ∇'. If φ and ψ are sections of E and X is a vector field on M, then

$$X\langle\varphi, \psi\rangle = \langle\nabla'_X\varphi, \psi\rangle + \langle\varphi, \nabla'_X\psi\rangle.$$

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