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SINGULARITIES OF 2-SPHERES IN 4-SPACE AND COBORDISM OF KNOTS[1]

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Consider an oriented 2-dimensional manifold m imbedded as a subcomplex in a triangulated oriented 4-dimensional manifold M in such a way that the boundary of m is contained in the boundary of M and the interior of m is contained in the interior of M. We will assume that M is a "piecewise linear manifold": that is, the star neighborhood of any point should be piecewise linearly homeomorphic to a 4-simplex. One can measure the local singularity of the imbedding at an interior point x of m as follows. Let N denote the star neighborhood of x in M. The boundary $S=\partial N$ of N is a 3-sphere with an orientation inherited from that of M, and $k=m\cap\partial N$ is a 1-sphere with an orientation inherited from that of m. The oriented knot type κ of the imbedding of k in S is called[2] the *singularity* of the imbedding at x. When k is of trivial type in ∂N we may say that *the singularity is* 0 or that x is a *non-singular point* or that m is *locally flat* at x. A surface m is called *locally flat* if it is locally flat at each of its points.

REMARK. The singularity of m at x is clearly a combinatorial invariant of M,m,x; that is it is not altered if we subdivide M rectilinearly. We do not know whether or not this singularity is a topological invariant, except in the special case of a locally flat point. The topological invariance of the concept of local flatness is easily proved, making use of Dehn's lemma, [12, §28(i)].

Of course the local singularity can also be measured at a boundary point x. In this case ∂N is a 3-cell, $m\cap\partial N$ is a 1-cell spanning it, and the singularity is a type of spanning 1-cell. In this paper we shall consider only imbeddings whose boundary points are all non-singular.

Since a singular point must be a vertex in any triangulation of the pair $m\subset M$ the singular points are always isolated. If m is compact (as it will be from now on) there can therefore be only a finite number of singular points. For the rest of this paper m will be a 2-sphere and M will be the 4-dimensional euclidean space R^4; that is, the 4-sphere punctured at ∞. The basic problem

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1) This paper follows our announcement [3]. We wish to express our thanks to C.H. Giffen for help in the revision.

2) These concepts are due to V.K.A. Guggenheim [5, §7. 32].