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A DUALITY THEOREM FOR HOMOGENEOUS MANIFOLDS OF COMPACT LIE GROUPS

NAGAYOSHI IWAHORI and MITSUO SUGIURA

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Introduction

In 1938, T. Tannaka [10] found a duality theorem for compact groups. The theorem was deepened by C. Chevalley [2] for compact Lie groups. In fact his theory establishes an intimate connection between compact Lie groups and reductive linear algebraic groups. More precisely, he associated an affine algebraic group G^c defined over $\boldsymbol{R}$ with each compact Lie group G and proved that G is canonically isomorphic to the subgroup G_R of all $\boldsymbol{R}$-rational points of G^c. The existence of this isomorphism amounts to the Tannaka duality theorem in this case. Conversely, for every reductive algebraic linear group G^*, there exists a compact Lie group G such that G^c is isomorphic to G^* (Proposition 1). The purpose of this paper is to prove a duality theorem for the homogeneous spaces of compact Lie groups which is analogous to Chevalley's theorem. For each homogeneous space $M=G/H$ of a compact Lie group G, we construct a complex affine algebraic set M^c which will be called the complexification of M and prove that the associated algebraic group G^c of G acts on M^c rationally and transitively (Proposition 3). Moreover the isotropy subgroup of G^c at the origin is identified with the associated algebraic group H^c of H (Proposition 3). This proves that the quotient space $M^c=G^c/H^c$ of a complex reductive algebraic group G^c over a complex reductive group H^c is an affine algebraic set (corollary to Proposition 3). This fact is proved by Borel and Harish-Chandra [1] when G^c is connected. There is a natural bijection of M onto the subset M_1 of M^c consisting of all real points of M^c. This is our duality theorem (Theorem 1). By introducing the notion of linear representation of a homogeneous space, this duality theorem can be formulated as a theorem of classical Tannaka-Chevalley type (Theorem 2). As the consequences of our duality theorem, we obtain the following results (Theorem 3).

1) The homogeneous space G/H of a compact Lie group G has the structure of a real affine algebraic set. 2) G/H has a faithful linear