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Series Representations of Certain Types of Arithmetical Functions

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1. Introduction. In this note we consider a special class of arithmetical functions which admit of seris expansion of the form (2.4). Two general theorems are proved in §2, the first being of representation-inversion type (Theorem 2.1), the second being an elementary asymptotic estimate for summatory functions (Theorem 2.2).

These two results are applied in §3 to functions arising from the generalized Euler function,

$$\phi_s(n) = \sum_{d\delta=n} \mu(d)\delta^s, \qquad \phi_1(n) = \phi(n), \tag{1.1}$$

$\mu(n)$ denoting the Möbius symbol, and the generalized Dedekind function,

$$\psi_s(n) = \sum_{d\delta=n} \mu^2(d)\delta^s, \qquad \psi_1(n) = \psi(n). \tag{1.2}$$

In particular, assuming $s>1$, we obtain in Theorem 3.2 estimates for the average order of $(\phi_s(n)/n^s)^k$, $(n^s/\phi_s(n))^k$, $(\psi_s(n)/n^s)^k$, $(n^s/\psi_s(n))^k$, where k is an arbitrary positive integer. These estimates are based on the series representations obtained in Theorem 3.1. Other combinations of $\phi_s(n)$ and $\psi_s(n)$ are also considered.

For a discussion of similar functions, corresponding to the case $s=1$ (excluded here), we mention Chowla [1] and Ward [5]. Their methods are quite different, however, from the one used in the present paper.

2. Two general theorems. The first theorem is based on the following well-known inversion formula for infinite series (cf. [4, Theorem 270]).

Lemma 2.1. *If $F(n)$ is an arithmetical function such that*

$$\sum_{r_1, r_2=1}^{\infty} |F(r_1 r_2)| \tag{2.1}$$

converges, then $F(n)$ may be represented in the form

$$F(n) = \sum_{r=1}^{\infty} g(nr), \tag{2.2}$$