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Remarks on the Equations of Evolution in a Banach Space

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§ **0. Introduction.** The contents of this paper consist of a slight extension of the previous paper [5] and some supplements to it. As in [5], we consider a certain type of the equations of evolution in a Banach space $\mathfrak{X}$:

$$(0.1) \qquad dx(t)/dt = (A(t)+B(t))x(t)+f(t)$$

and the associated homogeneous equation

$$(0.1') \qquad dx(t)/dt = (A(t)+B(t))x(t).$$

Here, $A(t)$ and $B(t)$ satisfy all the assumptions in [5] only replacing $\|\exp(tA(s))\| \leqq 1$ by $\|\exp(tA(s))\| \leqq M$, where M is a positive constant which is independent of t or s and generally greater than one. A necessary and sufficient condition that a closed operator generates a semi-group of bounded operators satisfying such an inequality was given by R. S. Phillips [3]. In [5], we assumed $M=1$ so that we were assured of the uniqueness of the solution of (0.1) by Theorem 1 of T. Kato [1]. But, we shall show the uniqueness in this paper without making such an assumption by examining the property of the fundamental solution $U(t, s)$ constructed in [5] a little closely. Note that it was unnecessary to assume $M=1$ in constructing $U(t, s)$ in [5].

In [5], we constructed the fundamental solution $U(t, s)$ first for the equation with $B(t)\equiv 0$, and then for the equation with $B(t)\not\equiv 0$ by a perturbation method. In this paper, we shall construct $U(t, s)$ directly even when $B(t)\not\equiv 0$ without using a perturbation method and show the further differentiability of the solution of (0.1) under the assumption that $A(t)$, $B(t)$ and $f(t)$ are sufficiently smooth, which was done only when $B(t)\equiv 0$ and $f(t)\equiv 0$ in [5]. In § 5, we shall give some remarks on generalized solutions in the sense of Solomiak [4], and finally in § 6, the case will be considered in which $\mathfrak{D}(A(t))$ changes smoothly with t in the sense of T. Kato [2].

§ 1. The uniqueness of the solution and some remarks.

Throughout this paper except in § 6, we assume that the operators