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On an Arcwise Connected Subgroup of a Lie Group

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It was recently proved that *an arcwise connected subgroup of a Lie group is a Lie subgroup* [1]. In this note a direct proof for it will be given.

Let A be an arcwise connected subgroup of an r-dimensional Lie group with the Lie algebra l, and we denote U_k a system of neighbourhoods of the identity e such that

$$U_1 \supset U_2 \supset \cdots$$

$$\bigcap_{k=1}^{\infty} U_k = e,$$

and by C_k the arcwise connected component of e in $U_k \cap A$.

We consider the directions $\overrightarrow{e, a_k}$ for $a_k \in C_k$, which converge to a limit direction Δ for a suitable sequence $\{a_k\}$. Let us denote by $X(\Delta)$ one of the corresponding infinitesimal transformations to Δ and by $\mathfrak{G}$ the aggregate of $X(\Delta)$'s.

For a one parameter subgroup $H_x = \{x\,;\ x = \exp \tau X,\ -1 \leqq \tau \leqq 1\}$ for $X \in \mathfrak{G}$, there exists a sequence $\{a_k\}$ so that $\overrightarrow{e, a_k}$ converge to the direction corresponding to X. That is for an arbitrarily small neighbourhood V [2] of e, there exist a pair of integers k and m [3] such that

$$(a_k)^j \subset H_x \cdot V, \qquad (a_k)^m \in (\exp X) \cdot V,$$

where $-m \leqq j \leqq m.$, Put $(a_k)^m = b(1)$ and $(a_k)^{-m} = b(-1)$. Now let us denote by γ_k the continuous curve which is drawn from e to a_k in U_k. Then it is possible to join $b(1)$ and $b(-1)$ by $\Gamma_x = \{(a_x)^j \gamma_k,\ -m \leqq j \leqq m\}$ in such a way that $\Gamma_x \subset H_x \cdot V$. Moreover we can introduce a parameter τ such that

$$\Gamma_x = \{b(\tau),\ -1 \leqq \tau \leqq 1\},\ b(\tau) \in (\exp \tau X) \cdot V.$$

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1) This theorm was proved by Iwamura, Hayashida, Minagawa and Homma when the Lie group is a vector group and by Kuranishi when it is semi simple. Kuranishi, using the above results, proved it for the general case, but the author obtained independently the present proof.

2) In this paper V or V' denotes arbitrarily or sufficiently small neighbourhood of the identity.

3) m depends upon a_k and X.