Chen, X. and Shen, Z.
Osaka J. Math.
40 (2003), 87–101

RANDERS METRICS WITH SPECIAL CURVATURE PROPERTIES

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(Received July 19, 2001)

1. Introduction

A Finsler metric on a manifold is a family of Minkowski norms on tangent spaces. There are several notions of curvatures in Finsler geometry. The flag curvature **K** is an analogue of the sectional curvature in Riemannian geometry. The distortion τ is a basic invariant which characterizes Riemannian metrics among Finsler metrics, namely, $\tau = 0$ if and only if the Finsler metric is Riemannian. The vertical derivative of τ on tangent spaces gives rise to the mean Cartan torsion **I**. The horizontal derivative of τ along geodesics is the so-called S-curvature **S**. The vertical Hessian of $(1/2)\mathbf{S}$ on tangent spaces is called the E-curvature. Thus if the S-curvature is isotropic, so is the E-curvature. The horizontal derivative of **I** along geodesics is called the mean Landsberg curvature **J**. Thus $\mathbf{J}/\mathbf{I}$ is regarded as the relative growth rate of the mean Cartan torsion along geodesics. We see how these quantities are generated from the distortion. Except for the flag curvature **K**, the above quantities are all non-Riemannian, namely, they vanish when F is Riemannian. See Section 2 for a brief discussion and [8] for a detailed discussion. In this paper, we will study a special class of Finsler metrics — Randers metrics with special curvature properties.

Randers metrics are among the simplest Finsler metrics, which arise from many areas in mathematics, physics and biology [1]. They are expressed in the form $F = \alpha+\beta$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form with $\|\beta\|_p := \sup_{\mathbf{y}\in T_pM} \beta(\mathbf{y})/\alpha(\mathbf{y}) < 1$ for any point p. Randers metrics were first studied by physicist, G. Randers, in 1941 [7] from the standard point of general relativity [1]. Since then, many Finslerian geometers have made efforts in investigation on the geometric properties of Randers metrics.

The shortest time problem on a Riemannian manifold also gives rise to a Randers metric. Given an object which can freely move over a Riemannian manifold (M, α), the object is pushed by a constant internal force $\mathbf{u}$ with $\alpha(\mathbf{u}) = 1$. We may assume that the object moves at a constant speed, due to friction. In this case, any path of shortest time is a shortest path of α. If there is an external force field $\mathbf{x}$ acting on the object with $\alpha(\mathbf{x}) < 1$, then any path of shortest time is a shortest path of the following

*The first author was supported by the National Natural Science Foundation of China (10171117).