Hattori, A. and Masuda, M.
Osaka J. Math.
40 (2003), 1–68

THEORY OF MULTI-FANS

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(Received May 1, 2001)

1. Introduction

The purpose of the present paper is to develop a theory of multi-fans which is an outgrowth of our study initiated in the work [27] on the topology of torus manifolds (the precise definition will be given later). A multi-fan is a combinatorial object generalizing the notion of a fan in algebraic geometry. Our theory is combinatorial by nature but it is built so as to keep a close connection with the topology of torus manifolds.

It is known that there is a one-to-one correspondence between toric varieties and fans. A toric variety is a normal complex algebraic variety of dimension n with a $(\mathbb{C}^*)^n$-action having a dense orbit. The dense orbit is unique and isomorphic to $(\mathbb{C}^*)^n$, and other orbits have smaller dimensions. The fan associated with the toric variety is a collection of cones in $\mathbb{R}^n$ with apex at the origin. To each orbit there corresponds a cone of dimension equal to the codimension of the orbit. Thus the origin is the cone corresponding to the dense orbit, one-dimensional cones correspond to maximal singular orbits and so on. The important point is the fact that the original toric variety can be reconstructed from the associated fan, and algebro-geometric properties of the toric variety can be described in terms of combinatorial data of the associated fan.

If one restricts the action of $(\mathbb{C}^*)^n$ to the usual torus $T = (S^1)^n$, one can still find the fan, because the orbit types of the action of the total group $(\mathbb{C}^*)^n$ can be detected by the isotropy types of the action of the subgroup T. Take a circle subgroup S of T which appears as an isotropy subgroup of the action. Then each connected component of the closure of the set of those points whose isotropy subgroup equals S is a T-invariant submanifold of real codimension 2, and contains a unique $(\mathbb{C}^*)^n$ orbit of complex codimension 1. We shall call such a submanifold a characteristic submanifold. If $M_1, \dots, M_k$ are characteristic submanifolds such that $M_1 \cap \cdots \cap M_k$ is non-empty, then the submanifold $M_1 \cap \cdots \cap M_k$ contains a unique $(\mathbb{C}^*)^n$-orbit of complex codimension k. This suggests the following definition of torus manifolds and associated multi-fans.

Let M be an oriented closed manifold of dimension $2n$ with an effective action of an n dimensional torus T with non-empty fixed point set M^T. A closed, connected, codimension two submanifold of M will be called characteristic if it is a connected component of the fixed point set of a certain circle subgroup S of T, and if it con-