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A THEOREM OF CALABI-MATSUSHIMA'S TYPE

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1. Introduction

In this note, we introduce the concept of the Einstein condition for a special type of conformally Kähler manifolds, called multiplier Hermitian manifolds in [7]. Then for such manifolds, an analogue of the theorems of Calabi [1] and Matsushima [8] will be proved.

For an n-dimensional compact connected Kähler manifold M with Kähler form ω_0, let $\mathcal{K}$ denote the set of all Kähler forms on M cohomologous to ω_0. We write each $\omega \in \mathcal{K}$ as

$$\omega = \sqrt{-1} \sum_{\alpha,\beta} g_{\alpha\bar{\beta}} dz^\alpha \wedge dz^{\bar{\beta}}$$

by using a system $(z^1, z^2, \dots, z^n)$ of holomorphic local coordinates on M. Let $G := \mathrm{Aut}^0(M)$ denote the identity component of the group of all holomorphic automorphisms of M. For a holomorphic vector field X on M, we put

$$\mathcal{K}_X := \{\, \omega \,;\, L_{X_\mathbb{R}} \omega = 0 \,\},$$

where $X_\mathbb{R} := X + \bar{X}$ denotes the real vector field on M associated to X. We say that X is *Hamiltonian* if in addition to $\mathcal{K}_X \neq \emptyset$, the holomorphic one-parameter subgroup

$$T := \{\, \exp(tX) \,;\, t \in \mathbb{C} \,\}$$

of G generated by X sits in the linear algebraic part of G, i.e., for each $\omega \in \mathcal{K}_X$, the holomorphic vector field X is expressible as

$$\mathrm{grad}^{\mathbb{C}}_\omega u_\omega := \frac{1}{\sqrt{-1}} \sum_{\alpha,\beta} g^{\bar{\beta}\alpha} \frac{\partial u_\omega}{\partial z^{\bar{\beta}}} \frac{\partial}{\partial z^\alpha}$$

for some real-valued smooth function $u_\omega \in C^\infty(M)_\mathbb{R}$ on M. Here u_ω is always normalized by the condition $\int_M u_\omega \, \omega^n = 0$. Throughout this note, we fix once for all a Hamiltonian holomorphic vector field $X \neq 0$ on M. In view of the moment map associated to the T-action on M, both $l_0 := \min_M u_\omega$ and $l_1 := \max_M u_\omega$ are independent