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THE STRUCTURE OF ALGEBRAIC EMBEDDINGS OF $\mathbb{C}^2$ INTO $\mathbb{C}^3$ (THE NORMAL QUARTIC HYPERSURFACE CASE. I)

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1. Introduction

A polynomial mapping $f : \mathbb{C}^n \to \mathbb{C}^m$ is called an *algebraic embedding* of $\mathbb{C}^n$ into $\mathbb{C}^m$ for $m > n \geq 1$ if f is injective and if the image of f is a smooth algebraic subvariety of $\mathbb{C}^m$. Let $\mathrm{Aut}(\mathbb{C}^n)$ be the group of algebraic automorphisms of $\mathbb{C}^n$. Here we consider the following conjecture:

Conjecture. Let $f : \mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$ be an algebraic embedding. Then f is equivalent to a linear embedding, that is, there exists an algebraic automorphism Φ of $\mathbb{C}^{n+1}$ such that $\Phi \circ f$ is a linear embedding.

For the case $n = 1$, Abhyankar-Moh [1] and Suzuki [16] (cf. [17]) showed that the conjecture is true. For the cases $n \geq 2$, the conjectures are still unsolved, however Russell [14], [15] has obtained some sufficient conditions for the conjectures to be true from a view point of ring theory. On the other hand, our approach in this paper is geometric and different from his. We use a method of compactifications of $\mathbb{C}^2$.

From now on, we will consider the case $n = 2$ only. Let $f : \mathbb{C}^2 \hookrightarrow \mathbb{C}^3$ be an algebraic embedding. We identify $\mathbb{C}^3$ with an affine part of the complex projective three-space $\mathbb{P}^3$ in the standard way. We denote by X_f the closure of the image of f in $\mathbb{P}^3$ and put $Y_f := X_f \setminus f(\mathbb{C}^2)$. By construction, we see that Y_f is a hyperplane section of X_f and that $X_f \setminus Y_f$ is biregular to $\mathbb{C}^2$, that is, (X_f, Y_f) is a *compactification of* $\mathbb{C}^2$. We call Y_f the *boundary* of the compactification. Our main purpose is, for the cases that the images of f are of low degree, to write down explicitly, up to affine transformations of $\mathbb{C}^3$, defining equations of the images and to construct explicitly algebraic automorphisms of $\mathbb{C}^3$ linearizing the defining equations. This explicit way is very important for us not only to obtain examples but also to find geometric invariants and inductive methods. In this direction, in our previous paper [12] (cf. [4], [5]), we have showed that the conjecture is true when the degree of the image is less than or equal to three. For the case of degree three, we needed a so-called *Nagata automorphism* (cf. [11]) to linearize some embedding.