

On Bott-Samelson K -cycles associated with symmetric spaces

By Shôrô ARAKI

(Received Nov. 16, 1962)

Introduction

In 1958, R. Bott and H. Samelson [8] defined the notion of K -cycles for every smooth complete Riemannian manifold M on which a compact connected Lie group K of isometries operates variationally completely, and showed that some K -cycles form a homology basis (mod 2 in general and integral in K -orientable cases) of some type of spaces of paths in M . They proved three kinds of variational completeness of K -actions related to symmetric spaces, and obtained many direct results.

In case K operates on itself or on its Lie algebra by adjoint actions, they determined moreover the integral cohomology of used K -cycles completely by making use of Cartan integers and applied it to several cohomological and homotopical problems of Lie groups [8], Chap. III.

The aim of the present work is to get an analogy (Theorems 2.10 and 6.4) of this for K -cycles associated with symmetric spaces, a partial result of which is used in determining the cohomology mod 2 of the compact exceptional group E_8 [3].

§ 1 is preliminaries about symmetric pairs, their Cartan subalgebras, restricted root systems, etc., including the definition of symmetric pairs of splitting rank. In § 2 we discuss basic properties of K -cycles associated with symmetric pairs. It is proved that every K -cycle associated with a symmetric pair is an iterated sphere bundle over a sphere (Cor. 2.5). Theorem 2.10 asserts that the cohomology rings mod 2 of K -cycles, associated with pairs (G, K) with simply connected G , are determined completely by Cartan integers of restricted roots. In §§ 3 and 4 we compute the number of connected components of centralizers in K of maximal tori and singular tori of symmetric pairs (G, K) with simply connected G . § 5 is a preparation for subsequent two sections.

In § 6 we discuss symmetric spaces of splitting rank. These behave themselves very similarly to compact Lie groups as symmetric spaces from homological point of view; for example, there holds an analogy (Prop. 6.3) of a well known result of J. Leray [10], Prop. 11.1. Here we prove Theorem 6.4 which asserts that the integral cohomology rings of K -cycles, associated with symmetric pairs (G, K) of splitting rank with simply connected G , are determined completely analogously to [8], Chap. III, Prop. 4.2, by Cartan integers of restricted roots.