Journal of Mathematics,
Osaka City University,
Vol. 13, No. 1.

Note on cohomology algebras of symmetric groups

Dedicated to Professor K. Shoda on his sixtieth birthday

by Minoru NAKAOKA

(*Received June 27, 1962*)

This is a continuation of the paper [3], and deals with the mod p cohomology algebra $H^*(S(m);\ Z_p)$ of the symmetric group $S(m)$ of degree m, where $1 \leqq m \leqq \infty$ and p is a prime. The author gave a basis for the homology module $H_*(S(m);\ Z_p)$ in [3]. In the present paper, we try to describe the diagonal homomorphism

$$d_*:\ H_*(S(m);\ Z_p) \longrightarrow H_*(S(m);\ Z_p) \otimes H_*(S(m);\ Z_p)$$

in terms of the basis, and by its conversion we derive some results on the cohomology algebra $H^*(S(m);\ Z_p)$. Throughout this paper a prime p is fixed.

1. Recapitulation.

For the convenience of the reader, the results which are proved in [2] and [3] are recapitulated in this section.

(A) Denote by $\lambda_m^n : S(m) \longrightarrow S(n)$ the natural inclusion map, where $m \leqq n$. Then, for any coefficient group G, the homomorphism $\lambda_{m*}^n : H_*(S(m); G) \longrightarrow H_*(S(n); G)$ induced by λ_m^n is a monomorphism and its image is a direct summand of $H_*(S(n); G)$; the homomorphism $\lambda_m^{n*}: H^*(S(n); G) \longrightarrow H^*(S(m); G)$ induced by λ_m^n is an epimorphism and its kernel is a direct summand of $H^*(S(n); G)$. If $q < (m+1)/2$ then $\lambda_{m*}^{m+1} : H_q(S(m); G) \longrightarrow H_q(S(m+1); G)$ and $\lambda_m^{m+1*} : H^q(S(m+1); G) \longrightarrow H^q(S(m); G)$ are isomorphisms.

(B) Let k be a field, and let $\mu:\ S(m) \times S(n) \longrightarrow S(m+n)$ denote a homomorphism defined by

$$\mu(\alpha\times\beta))(i) = \begin{cases} \alpha(i) & \text{if } 1 \leqq i \leqq m, \\ \beta(i-m)+m & \text{if } m < i \leqq m+n, \end{cases}$$

where $\alpha \in S(m)$ and $\beta \in S(n)$. Then, for elements $a \in H_i(S(m);\ k)$ and $b \in H_j(S(n);\ k)$ we define a product $ab \in H_{i+j}(S(m+n);\ k)$ by

$$ab = \mu_*(a \otimes b),$$

where $\mu_* : H_*(S(m);\ k) \otimes H_*(S(n);\ k) \longrightarrow H_*(S(m+n);\ k)$ is the homomorphism induced by μ. The product is bilinear, associative and (anti-) commutative. Denote by $S(\infty)$ the infinite symmetric group, *i.e.*, the direct limit of $\{S(m), \lambda_m^n\}$. Let $\lambda_m:\ S(m) \longrightarrow S(\infty)$ denote the natural inclusion. Then the rule

$$\lambda_{m*}(a)\ \lambda_{n*}(b) = \lambda_{m+n*}(ab)$$