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On root systems and an infinitesimal classification of irreducible symmetric spaces

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Introduction. The classification of real simple Lie algebras was given first by E. Cartan [2] in 1914. Though his first classification lacked in general theorems, Cartan himself [5] established in 1929 a general theorem suitable to simplify the classification. Then Gantmacher [6] in 1939 gave a simplified classification depending on Cartan's general theorem by making use of his theory on canonical representation of automorphisms of complex semi-simple Lie groups.

In his earlier papers [5, 3] Cartan established *a priori* a one-one correspondence between non-compact real simple Lie algebras and irreducible infinitesimal symmetric spaces (compact or non-compact), where "infinitesimal" means locally isomorphic classes. Hence the infinitesimal classification of irreducible symmetric spaces is the same thing as the classification of non-compact real simple Lie algebras.

Let $\mathfrak{g}$ be a real semi-simple Lie algebra and $\mathfrak{k}$ a maximal compact subalgebra of $\mathfrak{g}$. Then we have a Cartan decomposition

$$\mathfrak{g} = \mathfrak{k}+\mathfrak{p},$$

where $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ with respect to the Killing form. In the classical theories of classification of real simple Lie algebras due to E. Cartan and Gantmacher, one used a Cartan subalgebra $\mathfrak{h}_1$ of $\mathfrak{g}$ whose torus part $\mathfrak{h}_1 \cap \mathfrak{k}$ is maximal abelian in $\mathfrak{k}$, whereas certain geometric objects (such as roots, geodesics etc.) of symmetric spaces are related to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ whose vector part $\mathfrak{h} \cap \mathfrak{p}$ is maximal abelian in $\mathfrak{p}$ (cf., Cartan [4], Bott-Samelson [1] and Satake [7]). The two types of Cartan subalgebras mentioned above are not the same and even non-conjugate to each other in general. So it seems preferable to the author to have a classification theory by making use of the latter Cartan subalgebra, so as to connect it more closely with the theory of roots of symmetric spaces, and this will be developed in the present work.

If we denote by $\mathfrak{g}_C$ and $\mathfrak{h}_C$ the complexifications of $\mathfrak{g}$ and $\mathfrak{h}$ respectively, the conjugation of $\mathfrak{g}_C$ with respect to $\mathfrak{g}$ defines an involutive automorphism σ of the system $\mathfrak{r}$ of non-zero roots of $\mathfrak{g}_C$ relative to $\mathfrak{h}_C$. In §**1** are discussed some basic properties (Props. 1.1 and 1.3) of $\mathfrak{r}$ endowed with the invlution σ which are more or less known.

In §**2** we define root systems and σ-systems of roots in the abstract, and are sketched briefly some basic properties of them. Here is defined the notion that a σ-system of