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On some homogeneous spaces of classical Lie groups

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We shall, in this paper, give cellular decompositions of homogeneous spaces V_n, W_n and X_n of classical Lie groups $O(n)$, $U(n)$ and $Sp(n)$ by their diagonal subgroups respectively.

1. We denote by F one of three fields of real numbers R, complex numbers C or quaternion numbers Q, and by $d=d(F)$ the dimension of F over R; $d(R)=1$, $d(C)=2$ and $d(Q)=4$. Let F^n be a right vector space of dimension n over F and e_i $(i=1, \cdots, n)$ be the element of F^n whose i-th component is 1 and the others are 0. F^{n-1} is embedded in F^n as a vector subspace whose last component is 0.

Denote by $G(n)$ one of three classical Lie groups $O(n)$ (orthogonal group), $U(n)$ (unitary group) and $Sp(n)$ (symplectic group). $G(n)$ operates on F^n in the natural sense. $G(n-1)$ may be regarded as a subgroup of $G(n)$ by extending a point A of $G(n-1)$ to $G(n)$ by requirement that $Ae_n=e_n$.

The diagonal subgroup $D(n)$[1] of $G(n)$ is isomorphic to the product group $S^{d-1}\times\cdots\times S^{d-1}$ (n-fold), where S^{d-1} is the unit sphere in F which is a group. We define K_n to be $G(n)/D(n)$. Then we have $G(n-1)/D(n-1)=G(n-1)\times D(1)/D(n)\subset G(n)/D(n)$. Thus we have a sequence

$$K_1\subset K_2\subset \cdots \subset K_n\subset \cdots .$$

In the natural sense $G(n)$ operates on K_n, i.e. for $g\in G(n)$ and $a\in K_n$, we have $ga\in K_n$.

K_n is denoted by V_n, W_n or X_n, according as the field F is real, complex or quaternionic respectively.

Let Ω_{n-1} be the $d(n-1)$-dimensional projective space over F. If a point x of Ω_{n-1} has a representative $x=[x_1, \cdots, x_n]$, where $x_1, \cdots, x_n$ are, not all zero, in F, then the other representatives are $x=[x_1a, \cdots, x_na]$, where a is any non zero element of F. Hence we can choose a representative $x=[x_1, \cdots, x_n]$ such that $|x_1|^2+\cdots+|x_n|^2=1$. Now, if we define a mapping

$$\iota:\ \Omega_{n-1}\longrightarrow G(n)$$

by the formula

$$\iota([x_1, \cdots, x_n])=(\delta_{ij}-2x_i\bar{x}_j), \qquad i, j=1, \cdots, n,$$

1) In the case $G(n)=U(n)$, $D(n)$ is a maximal torus of $U(n)$.