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On the homology of classical Lie groups

By Ichiro YOKOTA

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1. Introduction

We shall give, in this paper, cellular decompositions of the classical Lie groups $SO(n)$, $SU(n)$ and $Sp(n)$. The important role is to give the primitive cells by making use of cross-sections from cells such that spheres $S^{n-1}=SO(n)/SO(n-1)$, $S^{2n-1}=SU(n)/SU(n-1)$ and $S^{4n-1}=Sp(n)/Sp(n-1)$ minus one point, respectively, to $SO(n)$, $SU(n)$ and $Sp(n)$. The cells of $SO(n)$ are closely connected with the real projective space P [7], [10] and the cells of $SU(n)$ are closely connected with the suspended space $E(M)$ of the complex projective space M [11]. The cells of $Sp(n)$, however, have no connection with the quaternion projective space directly.

In the classical Lie groups, the cup products and the Pontrjagin products are calculated rather simply : the Pontrjagin products of cells, fortunately, are cellular in the almost cases. As for the Steenrod's reduced powers, since these operations are calculated in the projective spaces P and M (and hence $E(M)$), we can calculate some reduced powers in $SO(n)$ and $SU(n)$. In the case of $Sp(n)$, we shall obtain the aim by researching the connections between $SU(2n)$ and $Sp(n)$.

The cellular decompositions of the classical Lie groups follow cellular decompositions of the Stiefel manifolds $V_{n,m}=SO(n)/SO(n-m)$, $W_{n,m}=SU(n)/SU(n-m)$, $X_{n,m}=Sp(n)/Sp(n-m)$ and some homogeneous spaces $F_n=SO(2n)/SU(n)$, $X_n=SU(2n)/Sp(n)$. We shall compute their homological properties by making use of their cell structures.

2. Notations

Let X be a finite cell complex and Γ a coefficient commutative ring with a unit. We denote by $H(X; \Gamma)$ (resp. $H^*(X; \Gamma)$) the homology group (resp. cohomology algebra) of X with coefficient ring Γ. If $f: X \to Y$ is a continuous mapping, we denote by ${}_\Gamma f_{\#}$ (resp. ${}_\Gamma f^{\#}$) the chain (resp. cochain) homomorphism and by ${}_\Gamma f_* : H(X; \Gamma) \to H(Y; \Gamma)$ (resp. ${}_\Gamma f^* : H^*(Y; \Gamma) \to H^*(X; \Gamma)$) the homomorphism (resp. algebraic homomorphism) induced by f respectively. Throughout this paper, Γ will be Z or Z_p.[1] According as Γ is Z or Z_p, ${}_\Gamma f_{\#}$ (resp. ${}_\Gamma f^{\#}$) and

1) Z is a free cyclic group with one generator. Z_p is a cyclic group of order p, where p is a prime integer.