Journal of the Institute of Polytechnics,
Osaka City University,
Vol. 1, No. 2, Series, A

On the Foundation of Orders in Groups

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(Received May 20, 1951)

0. Orderd groups, or o-groups, have been studied by G. Birkhoff, A. H. Clifford, H. Cartan, T. Nakayama, C. J. Everett and S. Ulam, J. von Neuman and others, while lattice ordered groups, or l-groups, also discussed by many mathematicians, G. Birkhoff and others.[1]

The present work is to establish the structure or order-buds (cf. below) in groups. This notion is closely conjugated with that of algebraic systems[2]; in other words, an order-bud is nothing but the modification of an algebraic order in groups.

1. We shall begin with some definitions. Let **G** be a group, e being its group identity, and P a subset of **G** with the following properties;

i) $e \in P$,

ii) $PP \subset P$.

We call such P *an order-bud in* **G**; in facts, we can define an order in **G**, $x \leqq y$ (P,l), x, $y \in G$, when $x^{-1} y \in P$, and another order, $x \leqq y (P,r)$, when $yx^{-1} \in P$.

The former, $\leqq (P.l)$, is called a, *left order* in G, while the latter, $\leqq (P,r)$, a *right order*.

(1. 1) If $x \leqq y(P,l)$ or $x \leqq y(P,r)$, then for all $t \in G$, $tx \leqq ty(P,l)$ or $xt \leqq yt(P,r)$ respectively.

It comes from the equalities $(tx)^{-1} \cdot \mathrm{ty} = x^{-1}t^{-1}ty = x^{-1}y$ and $yt \cdot (xt)^{-1} = ytt^{-1}x^{-1} = yx^{-1}$.

(1. 2) The set of all t such that $x \leqq t(P,l)$ or $x \leqq \mathrm{t(P,r)}$ coincides with $x \cdot P$ or $P \cdot x$ respectively.

We denote that $x \cdot P = P^l_x$, $P \cdot x = P^r_x$, then we have

(1. 3) $P^l_e = P^r_e = P$.

(1. 4) $y \in P^l_x$ implies $P^l_y \subset P^l_x$ and $y \in P^r_x$ implies $P^r_y \subset P^r_x$.

(1. 5) $a \cdot P^x_l = P^l_{ax}$, $P^r_x \cdot a = P^r_{xa}$,

If an order-bud P in **G** fulfils the further condition; for every $t \in G$,

iii) $tPt^{-1} \subset P$,

then we call P *normal*.

(1. 6) $P^l_a = P^r_a = a \cdot P = P \cdot a$ for normal P.

We put $P^l_a (= P^r_a) = P_a$ for normal P.

Let $\{P^\lambda\}_{\lambda \in \Lambda}$ be a family of order-buds in **G**, then the set-intersection of them

$$\cap_{\lambda \in \Lambda} P^\lambda$$