On a Necessary and Sufficient Condition of Metrizability

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It is well known that the second countability axiom is sufficient for a regular space to be metrizable, but it is not necessary. In this paper we shall show that some extension of the second countability axiom is necessary and sufficient for a regular space to be metrizable, and shall study some applications of this result.

Definition. Let $R$ be a topological space and $\{U_\beta | \beta \in B\}$ ($\alpha \in A$) be open coverings of $R$. We call $\{U_\alpha | \alpha \in A\}$ an open basis of $R$, when each open set $N$ of $R$ can be represented in the form

$$N = \sum_{U_\alpha \subset N} U_\alpha.$$

$\beta$-Countability Axiom. We say that $R$ satisfies $\beta$-countability axiom, when there exists an open basis of $R$ consisting of an enumerable number of nbd (=neighbourhood) finite open coverings $\{U_\alpha\}$.

It is an extension of the second countability axiom.

$\alpha$-Countability Axiom. We say that $R$ satisfies $\alpha$-countability axiom, when there exists a collection of an enumerable number of nbd finite coverings $\{U_n | n = 1, 2, \ldots\}$, $\{U_\beta | U_\beta = \{U_\alpha | \beta \in B\}\}$ such that for each pair of points $a, b \in R$, $a \neq b$, there exists $U_\alpha \in U_n : a \in U_\alpha, b \notin U_\alpha$ for some $n$.

We shall say that $\{U_\alpha\}$ satisfies the condition of $\alpha$-countability, when $\{U_\alpha\}$ has the above property.

Remark. The fact that $U_\alpha$ covers $R$ is not essential in $\alpha, \beta$-countabilities. For when $U_\alpha$ does not cover $R$, we may consider the covering $\{R, U_\alpha\}$ in the place of $U_\alpha$.

Theorem 1. In order that a regular space $R$ is metrizable, it is necessary and sufficient that $R$ satisfies the $\beta$-countability axiom.

Proof. Since the necessity is obvious from the theorem of A. H. Stone\(^1\), we prove only the sufficiency.

1. Let $R$ be a regular space satisfying the $\beta$-countability axiom.