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ON SOME SHARPLY T-TRANSITIVE SETS

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Let S_k be the symmetric group on a set $\Omega=\{1, 2, \cdots, k\}$ and t be an integer with $t\geqq 2$. A sharply t-transitive set G on Ω is a subset of S_k with the property that for every two ordered t-tuples $\alpha_1, \cdots, \alpha_t$ and $\beta_1, \cdots, \beta_t$ of elements in $\Omega(\alpha_i \neq \alpha_j, \beta_i \neq \beta_j$ for $i\neq j)$ there uniquely exists $g\in G$ which takes α_i into β_i: $(\alpha_i)g=\beta_i(i=1, \cdots, t)$. If $t=k-1$, G is S_k. So from now on we assume $t<k$. Although the sharply t-transitive groups were classified by Jordan and Zassenhaus (cf. [1]), it seems difficult to classify the sharply t-transitive sets. Now we define a distance d in S_k as follows: For two elements g_1 and g_2 in S_k,

$$d(g_1, g_2) = |\{\alpha\in\Omega\colon (\alpha)g_1 \neq (\alpha)g_2\}| .$$

Then (S_k, d) is a metric space and we have the following two propositions.

Proposition 1. *Let g be an element in a sharply t-transitive set G on $\Omega(|\Omega|=k)$ and $x_i(0\leqq i\leqq k)$ denote the number of elements $g'\in G$ satisfying $d(g, g')=k-i$. Then the following equality holds for $i=0, 1, \cdots, t-1$:*

$$x_i = \sum_{j=i}^{t-1}\binom{j}{i}\binom{k}{j}\{(k-j)(k-j-1)\cdots(k-t+1)-1\}(-1)^{j+i} .$$

In particular x_i's are uniquely determined independent of the choice of an element g in G.

Proof. Counting in two ways the number of the set $\{(g', (\alpha_1, \cdots, \alpha_i))\colon g'$ an element $\neq g$, $\{\alpha_1, \cdots, \alpha_i\}\subseteq\Omega$, $\alpha_u\neq\alpha_v$ for $u\neq v$, $(\alpha_j)g=(\alpha_j)g'$ for $j=1, \cdots, i.\}$ gives the following equality for $i=0, 1, \cdots, t-1$:

$$x_i+\binom{i+1}{i}x_{i+1}+\cdots+\binom{t-1}{i}x_{t-1} = \binom{k}{i}\{(k-i)(k-i-1)\cdots(k-t+1)-1\} .$$

Hence we have

$$M\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{t-1} \end{pmatrix} = \begin{pmatrix} \binom{k}{0}\{k(k-1)\cdots(k-t+1)-1\} \\ \binom{k}{1}\{(k-1)(k-2)\cdots(k-t+1)-1\} \\ \vdots \\ \binom{k}{t-1}\{(k-t+1)-1\} \end{pmatrix},$$