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Note on Lattice-Isomorphisms between Abelian Groups and Non-Abelian Groups

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The purpose of this note is to settle the problem of determining the groups lattice-isomorphic to abelian groups. This question was first put and studied by R. Baer. K. Iwasawa determined completely those finite groups and infinite groups with elements of infinite order whose lattices of subgroups are modular ($=$ *m-groups*), and determined the infinite m-group without elements of infinite order under the hypothesis that any m-group which has the lattice of subgroups of finite dimension is a finite group[1]. We shall call this *hypothesis* (A). So the only thing for us to do now is to find out whether non-abelian m-groups are lattice-isomorphic to abelian groups or not. In the case of finite groups this question was completely studied by A. W. Jones[2], and in the general case by R. E. Beaumont[3] to some extent.

We shall show in this note the following:

If G is a non-abelian infinite m-group and has no element of infinite order, under the hypothesis (A), similar theorems as those by Jones in the finite case hold, while if G has at least one element of infinite order, then there exists always an abelian group lattice-isomorphic to G.

We shall denote by $LC(G)$ and $L(G)$ the partially ordered set formed of all cyclic subgroups and the lattice formed of all subgroups of a group G respectively.

Definition. Let s, u and x be positive integers and α be an integral p-adic number such that $\alpha \equiv 1 \mod p^s (s \geqq 2$ if $p=2)$, then we define

$$(1) \qquad \varphi(\alpha, u, x) = \sum_{i=0}^{x-1} \alpha^{iu}.$$

When the value of α remains fixed, we shall write $\varphi(u, x)$ for $\varphi(\alpha, u, x)$. Let $\alpha = 1 + p^s\beta$, then $(1+p^s\beta)\cdot\varphi(1, u) = \varphi(1, u) + (1+p^s\beta)^u - 1$, hence $(1+p^s\beta)^u = \varphi(1, u)\, p^s\beta + 1$, and so

$$\begin{aligned}\varphi(u, x) &= \sum_{i=0}^{x-1} \{1+\varphi(1, u)\, p^s\beta\}^i \\ &= x + \frac{x!}{2!\,(x-2)!}\varphi(1, u)\, p^s\beta + \dots + \frac{x!}{r!\,(x-r)!}(\varphi(1, u)\, p^s\beta)^{r-1} + \dots \\ &\quad + \frac{x!}{x!}(\varphi(1, u)\, p^s\beta)^{x-1}.\end{aligned}$$

1) Iwasawa [1], [2], [3], cf. also Sato [1].
2) Jones [1].
3) Beaumont [1]. I could not see this paper.