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ON THE HOMOTOPY CLASSIFICATION OF 4-MANIFOLDS HAVING THE FUNDAMENTAL GROUP OF AN ASPHERICAL 4-MANIFOLD

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1. Introduction.

In this paper we shall study the homotopy type of closed connected oriented topological 4-manifolds M^4 with fundamental group isomorphic to $\Pi_1(Q)$, where Q is a fixed closed oriented aspherical 4-manifold. A standard example of such a manifold is the connected sum $M = Q\#M'$, where M' is an arbitrary simply-connected closed 4-manifold. In general, we shall always assume that M and Q are provided with CW-structures (up to homotopy) such that $M^{(3)} = M\backslash\overset{\circ}{D}{}^4$ and $Q^{(3)} = Q\backslash\overset{\circ}{D}{}^4$ (see for example [16], Lemma 2.9). Here the symbol $X^{(q)}$ denotes the q-skeleton of a CW-complex X as usual.

There are long outstanding conjectures concerning the topological structure of aspherical 4-manifolds (see for example [5]). One of these states that the Whitehead group of $\Pi_1(Q)$ is zero. So we can not assume in our case that homotopy equivalences are automatically simple.

Let $\Lambda = \mathbb{Z}[\Pi_1(Q)]$ be the integral group ring of $\Pi_1(Q)$ and $\mathrm{Out}(\Pi_1(Q))$ the outer automorphism group of $\Pi_1(Q)$, i.e., automorphisms modulo inner automorphisms.

Let $f : M \to Q$ be the classifying map of the universal covering. For this we shall prove the following result (see Section 3).

Theorem 1.1. *If f is of degree 1, then there is a homotopy equivalence of $M^{(3)}$ with $(Q\#M')^{(3)}$ for some simply-connected closed topological 4-manifold M'.*

As a consequence, $H_2(M;\Lambda)$ is Λ-free. In Section 2 we show that the classifying map $f : M \to Q$ is of degree 1 if and only if the k-invariant $k^3_M \in H^3(B\Pi_1;\Pi_2(M))$

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