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ON SOME ARITHMETICAL PROPERTIES OF ROGERS-RAMANUJAN CONTINUED FRACTION

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1. Introduction

Let $R(z)$ be the Rogers-Ramanujan continued fraction defined by

$$R(z) = 1 + \frac{z}{1\,+}\,\frac{z^2}{1\,+}\,\frac{z^3}{1\,+}\cdots \qquad (|z| < 1).$$

For $z = 1/q$ ($q \in \mathbb{N} - \{0, 1\}$), it is easy to transform $R(1/q)$ into the regular continued fraction

$$R(1/q) = 1 + \frac{1}{q\,+}\,\frac{1}{q\,+}\,\frac{1}{q^2\,+}\,\frac{1}{q^2\,+}\,\frac{1}{q^3\,+}\,\frac{1}{q^3\,+}\cdots$$

(see e.g. [9; 2.3]). Since this expansion is not ultimately periodic, $R(1/q)$ is not a quadratic number. More generally, as an application of a deep result of Nesterenko on modular functions [12], one can prove that $R(z)$ is transcendental for every algebraic number z ($0 < |z| < 1$) [5]. In this paper, we want to focus on the fact that $R(1/q)$ is not a quadratic number, and generalize this result in two directions.

First, we consider a more general Rogers-Ramanujan continued fraction

$$R(z; x) = 1 + \frac{zx}{1\,+}\,\frac{z^2x}{1\,+}\,\frac{z^3x}{1\,+}\cdots \qquad (|z| < 1).$$

Irrationality results on $R(z; x)$ for rational x and z are given in [11], [13], [14]. We will prove the following

Theorem 1. *Let $x = a/b \in \mathbb{Q}^*$ and let $z = 1/q$ with $q \in \mathbb{Z}$, $|q| \geq 2$. Suppose that $a^4 < |q|$. Then $R(1/q; a/b)$ is not a quadratic number.*

It should be noted that Lagrange's theorem on regular continued fractions cannot be applied here, because

$$R(1/q; a/b) = \frac{1}{qb/a\,+}\,\frac{1}{q\,+}\,\frac{1}{q^2b/a\,+}\,\frac{1}{q^2\,+}\,\frac{1}{q^3b/a\,+}\,\frac{1}{q^3\,+}\cdots$$