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Osaka J. Math.
37 (2000), 701–718

NUMERICAL BOUNDS OF CANONICAL VARIETIES

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(Received August 7, 1998)

0. Introduction

Let X be a minimal, complex, projective, Gorenstein variety of dimension n. We say that X is *canonical* if for some (any) desingularization $\sigma : Y \longrightarrow X$, the map associated to the canonical linear series $|K_Y|$ is birational.

We note K_X for the canonical divisor of X and $\omega_X = \mathcal{O}_X(K_X)$ the canonical sheaf. Let $p_g = h^0(X, \omega_X)$, $q = h^1(X, \mathcal{O}_X)$. There are several known bounds for K_X^n depending on p_g, the most general one being the bound $K_X^n \geq (n+1)p_g + d_n$ (d_n constant) given by Harris ([9]). Bounds including other invariants are known for canonical surfaces, $K_S^2 \geq 3p_g + q - 7$ ([12], [7]), and for surfaces and threefolds fibred over curves ([20], [25]).

In this paper we prove some results for canonical surfaces and threefolds. In the case of canonical surfaces there are some known results which show that under some additional hypotheses, the bound $K_S^2 \geq 3p_g + q - 7$ can be considerably improved (see Remark 2.2). We give here some other special cases (Remark 2.2) for which is not sharp and prove (Theorem 2.1) that, in fact, $K_S^2 = 3p_g + q - 7$ only if $q = 0$ whenever $p_g(S) \geq 6$.

Canonical surfaces with $K_S^2 = 3p_g - 7$ are known to exist and classified ([1]). Then we can hope that a good bound for canonical surfaces including the irregularity should be of type $K_S^2 \geq 3p_g + aq - 7$, $a > 1$. Since for $q = 1$ it is known ([16]) that $K_S^2 \geq 3p_g$, a should be 7, although unfortunately examples of low K_S^2 (with $q \geq 2$) are not known.

In the case of canonical threefolds we prove that $K_X^3 \geq 4p_g + 6q - 32$. In particular, we prove that the results of Ohno for canonical fibred threefolds are not sharp.

We use basically a result on quadrics containing irreducible varieties due to Reid ([21]) and several techniques originated in [24] and developed by Konno ([16], [17], [18]) for the study of the slope of fibred surfaces. In particular we include in an Appendix the dimension 3 version of the relative hyperquadrics method used by Konno in [18].

After this manuscript was written, the author was informed that Theorem 2.1 was known yet to K. Konno (unpublished).

Partially supported by CICYT PS93-0790 and HCM project n. ERBCHRXCT-940557