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Osaka J. Math.
37 (2000), 425–440

EXISTENCE OF SPIN STRUCTURES ON DOUBLE BRANCHED COVERING SPACES OVER FOUR-MANIFOLDS

Dedicated to the memory of Professor Katsuo Kawakubo

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(Received May 8, 1998)

1. Introduction

In this paper, we shall consider whether a double branched covering space over a given 4-manifold is spin or not. It is a fact known to experts that the double branched covering space $\widetilde{X}_{S^4}$ over the 4-sphere S^4 branched along a smoothly embedded connected closed surface F is spin if and only if F is orientable. However, the proof of this fact has never been published before as far as the author knows.

Hitchin [7] has shown that the double branched covering space $\widetilde{X}^q_{\mathbf{C}P^2}$ over the complex projective plane $\mathbf{C}P^2$ branched along a non-singular algebraic curve C of degree $2q$ is spin if and only if q is odd. Note that $\widetilde{X}^q_{\mathbf{C}P^2}$ is simply-connected [7, p. 22] and that $\widetilde{X}^3_{\mathbf{C}P^2}$ is a *K3* surface [5, p. 28].

The base spaces S^4 and $\mathbf{C}P^2$ of $\widetilde{X}_{S^4}$ and $\widetilde{X}^q_{\mathbf{C}P^2}$ respectively are both simply connected. The former is spin, and the latter is not. The total space $\widetilde{X}_{S^4}$ is spin when F is orientable, and the total space $\widetilde{X}^q_{\mathbf{C}P^2}$ is not necessarily spin, although the branch locus C is always orientable. Whether the double branched covering space $\widetilde{X}^q_{\mathbf{C}P^2}$ is spin or not depends on the homology class represented by the branch locus C in the base 4-manifold $\mathbf{C}P^2$.

Moreover, until now, there has been no example of a *spin* double branched covering space over a simply connected closed smooth 4-manifold branched along a smoothly embedded *non-orientable* surface.

In this paper, we consider a more general case. Let X be an oriented connected closed smooth 4-manifold with $H_1(X;\mathbf{Z}_2) = 0$ and $\widetilde{X}$ the double branched covering space over X branched along a connected closed surface F smoothly embedded in X. We consider whether $\widetilde{X}$ is spin or not.

Our result for this problem is the following.

Theorem 1.1. *Let X be a connected closed smooth* 4*-manifold with $H_1(X;\mathbf{Z}_2) = 0$ and $\widetilde{X}$ the double branched covering space over X branched along a connected closed surface F smoothly embedded in X. Then $\widetilde{X}$ is spin if and only if F is orientable and the modulo* 2 *reduction of $[F]/2 \in H_2(X;\mathbf{Z})$ coincides with the Poincaré*