ON HESSIAN STRUCTURES ON THE EUCLIDEAN SPACE
AND THE HYPERBOLIC SPACE

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1. Introduction

Let $M$ be a manifold with a flat affine connection $D$. A Riemannian metric $g$ on $M$ is said to be a Hessian metric if $g$ can be locally written $g = D^2 u$ with a local function $u$. We call such a pair $(D, g)$ a Hessian structure on $M$ and a triple $(M, D, g)$ a Hessian manifold ([5]). Hessian structure appears in affine differential geometry and information geometry ([1], [4]).

If $(D, g)$ is a Hessian structure on $M$, then in terms of an affine coordinate system $(x^i)$ with respect to $D$, $g$ can be expressed by $g = \sum_{ij} (\partial^2 u / \partial x^i \partial x^j) dx^i dx^j$. Since a Kähler metric $h$ on a complex manifold can be locally written $h = \sum_{i,j} (\partial^2 v / \partial z^i \partial \bar{z}^j) dz^i d\bar{z}^j$ with a local real-valued function $v$ in terms of a complex local coordinate system $(z^i)$, a Hessian manifold may be regarded as a real number version of a Kähler manifold. Thus we are interested in similarity between Kähler manifolds and Hessian manifolds.

Given a complex structure on a manifold, the set of Kähler metrics is infinite-dimensional. Similarly, given a flat affine connection, the set of Hessian metrics is infinite-dimensional. We next consider the converse situation. Given a Riemannian metric $g$, the set of almost complex structures $J$ that makes $g$ into a Kähler metric is finite-dimensional because $J$ is parallel with respect to the Riemannian connection. As a Hessian structure version of this, a question arises whether the set of flat affine connections that makes a given Riemannian metric into a Hessian metric is finite-dimensional. In this paper, we shall show that in the cases of the Euclidean space $(\mathbb{R}^n, g_0)$ and the hyperbolic space $(\mathbb{H}^n, g_0)$, the set of such connections is infinite-dimensional.

We prepare the terminology and notation. Let $(M, g)$ be a Riemannian manifold of dimension $\geq 2$ and $S^3(M)$ the space of all symmetric covariant tensor fields of degree 3 on $M$. We denote by $R$ and $\nabla$ the curvature tensor and the Riemannian connection, respectively. If $D$ is a flat affine connection of $M$ that makes $g$ into a Hessian metric, then the covariant tensor $T$ corresponding to $\hat{T} = D - \nabla$ by $g$ is an element of $S^3(M)$ satisfying $R^{\nabla + \hat{T}} = 0$ on $M$. Conversely, if the tensor $\hat{T}$ of type $(1, 2)$ corresponding to $T \in S^3(M)$ by $g$ satisfies $R^{\nabla + \hat{T}} = 0$ on $M$, then $D = \nabla + \hat{T}$ defines