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A CONSTRUCTION OF SURFACE BUNDLES OVER SURFACES WITH NON-ZERO SIGNATURE

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1. Introduction

Let Σ_g (respectively Σ_h) be a closed oriented surface of genus g (respectively h), where g (respectively h) is a non-negative integer. Let $\mathrm{Diff}_+\Sigma_h$ be the group of all orientation-preserving diffeomorphisms of Σ_h with C^∞-topology. A Σ_h-*bundle* over Σ_g (also called a *surface bundle* over a surface) is fiber bundle $\xi = (E, \Sigma_g, p, \Sigma_h, \mathrm{Diff}_+\Sigma_h)$ over Σ_g with total space E, fiber Σ_h, projection $p : E \longrightarrow \Sigma_g$ and structure group $\mathrm{Diff}_+\Sigma_h$. Our main concern is the signature $\tau(E)$ of the total space E of ξ.

It is easily seen that if ξ is a trivial bundle then $\tau(E) = \tau(\Sigma_g)\tau(\Sigma_h) = 0$. Chern-Hirzebruch-Serre [5] proved that if the fundamental group $\pi(\Sigma_g)$ of Σ_g acts trivially on the cohomology ring $H^*(\Sigma_h; \mathbb{R})$ of Σ_h then $\tau(E) = 0$.

Kodaira [12] and Atiyah [1] gave examples of surface bundles over surfaces with non-zero signature. For each pair (m, t) of integers $m, t \in \mathbb{Z}\,(m \geq 2, t \geq 3)$, Kodaira constructed a surface bundle $\xi = \xi(m, t)$ with

$$\begin{aligned} g &= m^{2t}(t-1)+1, \\ h &= mt, \\ \tau(E) &= \frac{4}{3}m^{2t-1}(t-1)(m^2-1). \end{aligned}$$

By setting $m = 2$ and $t = 3$, we obtain a surface bundle $\xi = \xi(2, 3)$ with $g = 129$, $h = 6$ and $\tau(E) = 256$. The total space E of the bundle $\xi = \xi(m, t)$ is an m-fold branched covering of $\Sigma_g \times \Sigma_t$ and its signature $\tau(E)$ can be calculated by using G-signature theorem(see [9] and [11]).

Meyer [16], [17] gave a signature formula for surface bundles over surfaces in terms of the *signature cocycle* τ_h, which is a 2-cocycle of the Siegel modular group $Sp(2h, \mathbb{Z})$ of degree h. Using the signature cocycle and Birman-Hilden's relations [3] of mapping class groups of surfaces, he showed that if $h = 1, 2$ or $g = 1$ then

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