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DECOMPOSITION THEOREM ON INVERTIBLE SUBSTITUTIONS

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0. Introduction

The decomposition theorem of automorphisms of free group is well known, and we mention the statement in the case of rank 2.

Theorem ([1]). *Let $G\{1,2\}$ be a free group generated by symbols* 1 *and* 2. *Then any automorphism of $G\{1,2\}$ is decomposed by three automorphisms*:

$$\alpha:\begin{cases}1\to 2\\ 2\to 1\end{cases},\quad \beta:\begin{cases}1\to 12\\ 2\to 1\end{cases},\quad \gamma:\begin{cases}1\to 1\\ 2\to 2^{-1}\end{cases}.$$

Recently Zhi-Xiong Wen and Zhi-Ying Wen give the decomposition theorem of invertible substitutions of rank 2, where we say an automorphism σ is an invertible substitution if words $\sigma(1)$ and $\sigma(2)$ consist of the symbols 1 or 2.

Theorem ([2]). *Any invertible substitution is generated by three invertible substitutions*:

$$\alpha:\begin{cases}1\to 2\\ 2\to 1\end{cases},\quad \beta:\begin{cases}1\to 12\\ 2\to 1\end{cases},\quad \delta:\begin{cases}1\to 21\\ 2\to 1\end{cases}.$$

In this paper we give a simple proof of the theorem and a geometrical charactarization of invertible substitutions.

1. Proof of the theorem

Let us introduce the canonical homomorphism $\mathbf{f}: G\{1,2\}\to \boldsymbol{Z}^2$ as follows:

$$\mathbf{f}(i^{\pm 1}) := \pm \boldsymbol{e}_i,\quad i=1,2$$
$$\mathbf{f}(W) := \mathbf{f}(s_1)+\mathbf{f}(s_2)+\cdots+\mathbf{f}(s_k)\quad \text{for}\quad W=s_1s_2\cdots s_k\in G\{1,2\}$$

where $\{\boldsymbol{e}_1,\boldsymbol{e}_2\}$ be canonical basis in $\boldsymbol{R}^2$. Then we know the following property.