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A SYMBOLIC CALCULUS FOR PSEUDO DIFFERENTIAL OPERATORS GENERATING FELLER SEMIGROUPS

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1. Introduction

It is well-known that the generator A of a time-homogenous Markov process in $\mathbb{R}^n$ is typically given by a Lévy-type operator

$$A\varphi(x) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2\varphi(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial\varphi(x)}{\partial x_i} + c(x)\varphi(x)$$

$$(1.1) \qquad + \int_{\mathbb{R}^n\setminus\{0\}} \left(\varphi(x+y) - \varphi(x) - \frac{\langle y, \nabla\varphi(x)\rangle}{1+|y|^2}\right)\mu(x,dy), \quad \varphi \in C_0^\infty(\mathbb{R}^n).$$

This follows immediately from the fact that the generator of a transition semigroup satisfies the positive maximum principle, i.e. for any φ in the domain of the generator and $x_0 \in \mathbb{R}^n$ such that $\varphi(x_0) = \sup_{x\in\mathbb{R}^n}\varphi(x) \geq 0$ we have $A\varphi(x_0) \leq 0$ and by a result of Ph. Courrège [4] which characterizes the operators satisfying the positive maximum principle as operators of type (1.1). But Courrège gave also another equivalent representation of this class of operators as pseudo differential operators

$$(1.2) \qquad A\varphi(x) = -p(x,D)\varphi(x) = -\int_{\mathbb{R}^n} e^{i(x,\xi)}p(x,\xi)\cdot\hat{\varphi}(\xi)\,đ\xi, \quad \varphi \in C_0^\infty(\mathbb{R}^n),$$

defined by a symbol $p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ having the crucial property that for fixed $x \in \mathbb{R}^n$ the function $p(x,\cdot)$ is a continuous negative definite function (see section 2 for the definition). Such symbols we briefly call negative definite symbols. Here $\hat{\varphi} = \int_{\mathbb{R}^n} e^{-i(x,\xi)}\varphi(x)\,dx$ denotes the Fourier transform and $đ\xi = (2\pi)^{-n}\,d\xi$. Conversely, if the symbol is a continuous negative definite function for every fixed $x \in \mathbb{R}^n$ then the operator $-p(x,D)$ satisfies the positive maximum principle on $C_0^\infty(\mathbb{R}^n)$.

The relation between (1.1) and (1.2) is given by the Lévy-Khinchin formula, see [2], which represents the continuous negative definite functions $p(x,\cdot)$ (for fixed x) in terms of the coefficients $a_{ij}(x)$, $b_i(x)$, $c(x)$ and the Lévy-measures $\mu(x,dy)$ of (1.1). In this paper we focus on the representation (1.2) as a pseudo differential operator and look for conditions purely in terms of the symbol $p(x,\xi)$ implying that the operator $-p(x,D)$ actually generates a Markov process. We are interested in particular in the