Borde, P. and Peraire, Y.
Osaka J. Math.
35 (1998), 771–787

AN APPROACH OF THE THEORY OF INTEGRATION BY THE MEAN OF BIASED TEST FUNCTIONS

PASCAL BORDE and YVES PERAIRE

(Received October 18, 1996)

Introduction

In [7], O. Loos has given, in the framework of IST (the Nelson's version of non standard analysis) a definition of the Lebesgue integral of a standard function f defined on a standard compact interval of $\mathbb{R}$ using a sort of approximation of f by steps functions. The advantage of this approach is that it reduces the Lebesgue integral to the level of difficulty of the ordinary Riemann integral. Other nonstandard theories on integration exist.

The best known and universally used nonstandard formulation has been given, in the Robinson approach of nonstandard analysis, by P. Loeb in the book of Hurd and Loeb ([4]).

The theory of Pierre Cartier and Yvette Perrin [2], is in the spirit of the phenomenological approach of mathematics of Vopenka ([18]); it is very clear and use only calculus of hyper-finite sums (finite but nonstandard). We can also say that the thought process of Loos is not so far from the classical approach of Kurzweil-Henstock in [3], with gauge functions and its nonstandard improvement by J. Mawhin ([10]).

Our aim is to generalize the Loos's ideas to non standard functions defined on any intervals of $\mathbb{R}^N$ (N possibly infinitely large). The framework of this paper is Relative Set Theory (see [13],...,[17]); we describe this theory and some results in the annex at the end of the article.

Let us first give some notations and definitions.

Let N be a natural number (possibly infinitely large), we call *interval* of $\mathbb{R}^N$ a cartesian product of N intervals of $\mathbb{R}$. Let $a = (a_1, \ldots, a_N)$ and $b = (b_1, \ldots, b_N)$ two points of $\mathbb{R}^N$, we denote respectively by $[a,b]$, $[a,b[$, $]a,b]$ and $]a,b[$ the intervals $\prod_{k=1}^N [a_k, b_k]$, $\prod_{k=1}^N [a_k, b_k[$, $\prod_{k=1}^N]a_k, b_k]$ and $\prod_{k=1}^N]a_k, b_k[$. Let $P = \prod_{k=1}^N I_k$ be an interval of $\mathbb{R}^N$ with $\overline{I_k} = [a_k, b_k]$ and let for all k, $\{x_1^k, \ldots, x_{n_k}^k\}$ be a subdivision of $\overline{I}_k$ with $a_k = x_1^k < \ldots < x_{n_k}^k = b$. We define a step function on P as a function which is constant on each open interval of $\prod_{k=1}^N]x_{j_k}^k, x_{j_{k+1}}^k[$, $j_k \in \{1, \ldots, n_k\}$ and we denote by $\mathcal{E}(P)$ the set of all step functions on P.

<u>Topology on the sets of intervals</u>: The reader is refered to [16] for more details about