

GENERALIZATIONS OF THEOREMS OF FULLER

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Let R be a right artinian ring and e a primitive idempotent of R . In [6, Corollary 3.2 and Theorem 3.4] (also see Anderson-Fuller [1, Theorem 31.3].) K. Fuller showed that the following conditions are equivalent.

- (1) eR is an injective right R -module.
- (2) There exists a primitive idempotent f of R such that
 (2*) $S(eR) \cong T(fR)$ and $S(Rf) \cong T(Re)$, where $S(M)$ and $T(M)$ denote the socle and the top of M , respectively.
- (3) There exists a primitive idempotent f of R such that
 (3l) $\ell_{eR}(r_{Rf}(I)) = eI$ for each left ideal I , and
 (3r) $r_{Rf}(\ell_{eR}(K)) = Kf$ for each right ideal K of R , where $r_{Rf}(I) = \{a \in Rf \mid Ia = 0\}$ and $\ell_{eR}(K) = \{b \in eR \mid bK = 0\}$.

Let R be a semiprimary ring and e and f primitive idempotents of R . Then (eR, Rf) is called an i -pair in [3] if the above condition (2*) is satisfied. In [3, Theorem 1, Proposition 4 and Corollary 1], Y. Baba and K. Oshiro extended these results to semiprimary rings to show the following statements.

- (a) If R is a semiprimary ring, then the condition (1) is satisfied if and only if both (2) and (3r) are satisfied.
- (b) If R is a semiprimary ring satisfying (2) and the condition (*) below, then (1) is satisfied.
- (*) The lattice $\{r_{Rf}(X) \mid X \subseteq eR\}$ satisfies the ascending chain condition. Moreover, in [3, Theorem 2], they showed the following statement (c).
- (c) If R is a semiprimary ring and (eR, Rf) is an i -pair for primitive idempotents e and f of R , then the following are equivalent.
 - (c1) Rf is artinian as a right fRf -module.
 - (c2) eR is artinian as a left eRe -module.
 - (c3) eR is an injective right R -module and Rf is an injective left R -module.

In this note, for a right R -module M with $S(M) \cong T(fR)$ and $P = \text{End}M$, we consider a pair $({}_P M, Rf{}_f Rf)$ instead of an i -pair $(eRe eR, Rf{}_f Rf)$ and give generalizations of the results (a), (b) and (c) above (in Sections 1 and 2). In particular, in Section 1, for a module N_Q , we give some properties for the pair $({}_P M, N_Q)$, which are very similar to Theorem 1.1 in Morita-Tachikawa [11]. Moreover, in Section 3, by applying results obtained in Sections 1 and 2, we give elementary proofs of