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# A SIMPLE PROOF OF THE RATIO ERGODIC THEOREM

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## 1. Introduction

Throughout the development of ergodic theory, attention has been devoted by many authors, beginning with the classical struggles of Birkhoff and von Neumann, to proofs of different forms of ergodic theorems. Recenttly, a standard principle has begun to emerge (see [6], but also [4], [5]). The aim of this short note is to apply the principle to obtain a proof of Hopf's ratio ergodic theorem ([1]).

**Fundamental Lemma.** *Let $(a_n)_{n=0,1,\cdots}$ and $(b_n)_{n=0,1,\cdots}$ be sequences of non-negative real numbers for which there exists a positive integer $M$ such that for any $n=0,1,\cdots$ there exists an integer m with $1 \le m \le M$ satisfying that*

$$\sum_{0\le i<m} a_{n+i} \ge \sum_{0\le i<m} b_{n+i}.$$

*Then for any integer $N$ with $N > M$,*

$$\sum_{0\le n<N} a_n \ge \sum_{0\le n<N-M} b_n.$$

Proof. The proof of the Fundamental Lemma is easy. By the assumption, we can take integers $0=m_0<m_1<\cdots<m_k<N$ with $m_{i+1}-m_i \le M \quad (i=0,1,\cdots,k-1)$ and $N-m_k<M$ such that

$$\sum_{m_i\le n<m_{i+1}} a_n \ge \sum_{m_i\le n<m_{i+1}} b_n.$$

for any $i=0,1,\cdots,k-1$. Then, by adding these inequalities, we have

$$\sum_{0\le n<N} a_n \ge \sum_{0\le n<m_k} a_n \ge \sum_{0\le n<m_k} b_n \ge \sum_{0\le n<N-M} b_n.$$

We apply this Lemma to prove the Ratio Ergodic Theorem [8]. Let $(\Omega,\mathcal{B},\mu)$ be a $\sigma$-finite measure space and $T:\Omega\to\Omega$ be a measure preserving transformation.