Zucconi, F.
Osaka J. Math.
34 (1997), 411–428

SURFACES WITH CANONICAL MAP OF DEGREE THREE AND $K^2 = 3p_g - 5$

FRANCESCO ZUCCONI*

(Recieved February 19, 1996)

Introduction

The canonical map of a nonsingular variety X of dimension n, $X \overset{\phi_{K_X}}{\longrightarrow} \boldsymbol{P}^{p_g-1}$, is the rational map given by $x \mapsto (s_1(x), \cdots, s_{p_g}(x))$ where $\langle s_i \rangle_{i=1,\cdots,p_g}$ is a basis of $H^0(X, \mathcal{O}_X(K_X))$ and where K_X, the so called canonical divisor, is a divisor such that $\mathcal{O}_X(K_X)$ is the sheaf of holomorphic n-forms. Let $\phi_{K_X}(X) = \Sigma$ be the image. If we assume $\dim \Sigma = n$ then there is a natural number $d = \deg \phi_{K_X}$ associated to K_X.

If $n = 1$ then d can only be 1 or 2 and $d = 1$ is the general case. The special case $d = 2$ occurs, and, by this feature, admits a very explicit description: in fact $d = 2$ if and only if X is a hyperelliptic curve.

If $n = 2$ Castelnuovo proved that if $K_X^2 < 3p_g - 7$ then $d = 2$ and Σ is a ruled surface, while if $K_X^2 = 3p_g - 7$ then $d = 1$ or $d = 2$ and Σ is a ruled surface. He also classified surfaces with $K_X^2 = 3p_g - 7$ and $d = 1$, (see [1] for a modern reference). Since then the theory of the canonical map of surfaces has been extensively studied by several authors; here we can quote [20], [18], [9], [3], [21], [15]. However, the case $d = 3$ is not yet well understood. The initial idea, due to Castelnuovo, to study the case $d = 3$ was to consider a fibration of X on a smooth curve B, $f : X \to B$, such that the canonical linear system $| K_X |$ induces a g_3^1 on the fibers of f. In fact, following this idea, Pompilj proved that if $d = 3$ and $K_X^2 = 3p_g - 6$ then $q = \dim_{\mathbb{C}} H^1(X, \mathcal{O}_X(K_X)) = 0$ and $(p_g, K_X^2) = (3,3), (4,6)$ or $(5,9)$. He also classified these surfaces completely ([18]). In the seventies Horikawa rediscovered these surfaces except for the case $p_g = 5$ ([12], [13]). In [14] Konno gives a detailed classification of surfaces with $K_X^2 = 3p_g - 6$. In particular he considers the case $d = 3$ and $p_g = 5$. We also know that if $d = 3$ and $q > 0$ then $K_X^2 \geq 3p_g - 4$ [7, Proposition 5.1]. Moreover by [21, Theorem 2] we know that $p_g \leq 9$ if $K^2 = 3p_g - 5$ and $d = 3$. Thus the problem of classifying surfaces with $K_X^2 = 3p_g - 5$ and $d = 3$ arises very naturally. In this paper we show that the line $K^2 = 3p_g - 5$ gives rise to two families which we completely described. One of them ($K^2 = 7$, $p_g = 4$, $\deg\phi_{|K_X|} = 3$) is of a certain interest for two reasons:

*Research carried out under the ECHCM project AGE (Algebraic Geometry in Europe), contract number ERBCHRXCT 940557 and C.N.R. 201.01.120