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DIRECT SUM OF LOCAL MODULES WITH EXTENDING FACTOR MODULES

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1. Introduction

Rings whose cyclic modules are continuous have been studied by Jain and Mohamed [9]. These rings are semiperfect rings. Semiperfect rings whose cyclics are π-injective (extending) are studied by Goel and Jain [6] (Vanaja [14]). We call a module an FE module if every factor module is extending. It was proved in [14] that for a semiperfect ring R, R_R is FE if and only if R_R is extending and every factor module of $R/Soc\,R$ is π-injective. One can easily extend the above result to modules M which are projective and semiperfect in $\sigma[M]$. In this case M is a direct sum of local modules. We extend the above result for any module M which is a direct sum of local modules.

The proof in the case when M is semiperfect and projective in $\sigma[M]$ heavily depends on the fact that M is a direct sum of locals with local endomorphism ring and this decomposition of M complements direct summands. Some sufficient conditions for a decomposition of a module M as a direct sum of locals to complement summands are proved in Section 4.

In Section 5 some important properties of an FE module which is a direct sum of two local modules are obtained. In Section 6 FE modules which are direct sum of local modules are considered. We do not assume that M is projective in $\sigma[M]$ or that the endomorphism ring of these local modules are local. We show that if $M = \bigoplus_{i\in I} M_i$ is an FE module, where each M_i is a local module, then this decomposition complements summands and any factor module of M is isomorphic to $\bigoplus_{i\in I} M_i/X_i$, for some $X_i \subseteq M_i$ (6.2). Our main theorem (6.3) is as follows.

Let $M = \bigoplus_{i\in I} M_i$, where each M_i is a local module. Then the following are equivalent:

(a) $\bigoplus_{i\in I} M_i/X_i$ is uniform-extending, for all $X_i \subseteq M_i$;
(b) $\bigoplus_{i\in I} M_i/X_i$ is extending, for all $X_i \subseteq M_i$;
(c) every factor module of M is extending;
(d) every factor module of M is uniform-extending;
(e) M is uniform-extending and $\bigoplus_{i\in I}(M_i/Soc\,M_i)/Y_i$ is π-injective, for all $Y_i \subseteq M_i/Soc\,M_i$;