ON REGULAR RINGS
WHOSE CYCLIC FAITHFUL MODULES
CONTAIN GENERATORS

HIROSHI YOSHIMURA

1. Introduction

The present paper may be considered as a continuation of [6], in which we studied nonsingular rings R satisfying the condition (C^*) that every cyclic faithful right R-module is a generator for the category Mod-R of all right R-modules. We proved in [6] that a (von Neumann) regular ring satisfies (C^*) if and only if it is isomorphic to a finite direct product of an abelian regular ring and full matrix rings over self-injective abelian regular rings (c.f. [4]). Concerning this, we note that there exists a regular ring R over which, although R fails to satisfy the condition (C^*), yet every cyclic faithful right R-module "*contains*" a submodule which is a generator for Mod-R. For instance, choose a division ring D_n containing a division subring E_n for $n = 1, 2, \ldots$, let k (≥ 2) be an integer, and let S_n and T_n be the rings of all $k \times k$ matrices over D_n and E_n, respectively, for $n = 1, 2, \ldots$. Now, consider the regular ring R which consists of all sequences $(x_n) \in \prod_{n=1}^{\infty} S_n$ such that $x_n \in T_n$ for all but finitely many n. Then, R satisfies (C^*) only when $D_n = E_n$ for all but finitely many n (see [4], or [6]), whence in case E_n is properly contained in D_n for infinitely many n, the ring R does not satisfy (C^*). However, it is shown that every cyclic (finitely generated) faithful right module over the ring R actually contains a generator. In fact, as will be noted in Example 3(2) of §3, the full matrix rings over any continuous abelian non-self-injective regular rings do not satisfy (C^*), but every cyclic (finitely generated) faithful module over the rings does contain a generator. This shows that the condition (C^*) is not equivalent to the one (C) that every cyclic faithful right module contains a submodule which is a generator.

In this paper, we shall consistently investigate regular rings satisfying the condition (C) above, and determine their structure. Section 2 is devoted to preliminary results on regular rings R satisfying (C), part of which would be derived from their more general property that every cyclic faithful right R-module is co-faithful. In Section 3 we shall present our main result (Theorem A) which asserts that the regular rings satisfying the condition (C) are precisely the finite direct products of abelian regular rings and full matrix rings over abelian regular rings S such that every finitely generated faithful right S-submodule of the maximal quotient ring