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EQUIVARIANT ALGEBRAIC VECTOR BUNDLES OVER A PRODUCT OF AFFINE VARIETIES

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0. Introduction

Let G be a reductive complex affine algebraic group and Z a complex affine G-variety with a G-fixed base point $z_0 \in Z$. Throughout this paper, the base field is the field $\boldsymbol{C}$ of complex numbers. Let Q be a G-module. We denote by $\mathrm{Vec}_G(Z,Q)$ the set of algebraic G-vector bundles over Z whose fiber at z_0 is Q and by $\mathrm{VEC}_G(Z,Q)$ the set of G-isomorphism classes in $\mathrm{Vec}_G(Z,Q)$. We denote by $[E]$ the isomorphism class of $E \in \mathrm{Vec}_G(Z,Q)$.

There are many interesting problems concerning $\mathrm{VEC}_G(Z,Q)$, especially when the base space Z is a G-module P. One of them is the Equivariant Serre Problem, which asks whether $\mathrm{VEC}_G(P,Q)$ is the trivial set consisting of the isomorphism class of the product bundle $P \times Q$. When G is trivial, the Quillen-Suslin Theorem says that $\mathrm{VEC}_G(P,Q)$ is the trivial set. More generally, Masuda-Moser-Petrie [9] recently have shown that $\mathrm{VEC}_G(P,Q)$ is trivial for any abelian group G. However, when G is not abelian, $\mathrm{VEC}_G(P,Q)$ is non-trivial in general. Schwarz [13] (see Kraft-Schwarz [5] for details) first presented counter examples to the Equivariant Serre Problem by proving that $\mathrm{VEC}_G(P,Q) \cong \boldsymbol{C}^p$ when the algebraic quotient space $P/\!/G$ is one dimensional i.e. isomorphic to affine line $\boldsymbol{A}$. When $\dim P/\!/G \geq 2$, there are many non-trivial examples of $\mathrm{VEC}_G(P,Q)$ ([11], [4]) but it remains open to classify elements in $\mathrm{VEC}_G(P,Q)$ in general.

The results of [13] extend to the case where the base space is a weighted G-cone with smooth one dimensional quotient (for a precise definition, see §1; a G-module with one dimensional quotient is an example of such a cone):

Theorem A ([8]). *Let X be a weighted G-cone with smooth one dimensional quotient and Q be a G-module. Then $\mathrm{VEC}_G(X,Q) \cong \boldsymbol{C}^p$ for a non-negative integer p. Moreover, there is a G-vector bundle $\mathfrak{B}$ over $X \times \boldsymbol{C}^p$ such that the map $\boldsymbol{C}^p \ni z \mapsto [\mathfrak{B}|_{X \times \{z\}}] \in \mathrm{VEC}_G(X,Q)$ gives a bijection.*

Masuda-Petrie have made the following observation. Let X and p be as above and Y an irreducible affine variety with trivial G-action. We denote by $\mathrm{Mor}(Y,\boldsymbol{C}^p)$ the set of morphisms from Y to $\boldsymbol{C}^p$. Then there is a map