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ON UPPERBOUNDS OF VIRTUAL MORDELL-WEIL RANKS

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0. Introduction

0.0. Let $f: X \to C$ be a relatively minimal fibration of curves of genus $g \geq 1$ over a smooth projective curve C of genus b defined over an algebraically closed field k. Let $K = k(C)$ be the field of rational functions on C. In the theory of Mordell-Weil lattices due to Shioda (cf. [17], [18]) the following conditions are assumed:

(0.1) (i) *f admits a global section (O) as zero-section,*
(ii) *K/k-trace of the Jacobian J_F of the generic fibre F/K of f is trivial.*

Under these conditions the Mordell-Weil group $J(K)$ of K-rational points of J is finitely generated. The rank r of its free part is called the Mordell-Weil rank. We shall be concerned with characteristic zero case (in this case the second assumption in (0.1) is equivalent to $q(X) = b$). In [14, Theorem 1.3] an upperbound of r via the invariants of f is given. In particular, for the case of rational surfaces X it was shown in a joint paper ([15]) that $r \leq 4g+4$. Moreover the structure of fibrations with maximal rank $r = 4g+4$ and the structure of corresponding Mordell-Weil lattices are completely determined in [15] (a such fibration is obtained as a blowing up of a linear pencil of hyperelliptic curves on a Hirzebruch surface Σ_e with $0 \leq e \leq g$ $(g \geq 2)$).

In this note we consider a similar problem for locally non-trivial fibrations, not necessarily satisfying conditions (0.1). Let $\mathrm{NS}(X)$ be the Néron-Severi group of X. Then $\mathrm{NS}(X)/torsion$ admits the lattice structure with the intersection pairing. Hodge's index theorem asserts that its signature is $(1, \rho-1)$, where $\rho := \mathrm{rank}\,\mathrm{NS}(X)$ is the Picard number of X.

DEFINITION 0.2 (cf. [11]). The virtual Mordell-Weil rank r of f is defined to be the rank of the essential sublattice of the Néron-Severi lattice (cf. [17], [18]), i.e.,

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