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ON UNIT-REGULAR RINGS SATISFYING S-COMPARABILITY

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1. Preliminaries and notations

In [2] and [3], we studied the properties of unit-regular rings satisfying the comparability axiom. In this paper, we shall investigate unit-regular rings satisfying s-comparability which is a generalized notion of the comparability axiom. In section 2, we shall show that these rings have the property (DF), that is, $P \oplus Q$ is directly finite for every two directly finite projective modules P and Q. In section 3, we shall obtain a criterion of direct finiteness of projective modules over these rings (Proposition 4 and Theorem 7). Using this result, we can determine the types of directly finite projective modules and classify the family of all unit-regular rings satisfying s-comparability into three types; Types A, B and C (Theorem 12). In section 4, we shall give the ideal-theoretic characterization for Types A, B and C (Theorems 14, 15 and 16).

Throughout this paper, R is a ring with identity and all modules are unital right R-modules.

NOTATION. If M and N are R-modules, then the notation $N \lesssim M$ (resp. $N \lesssim \oplus\ M$) means that N is isomorphic to a submodule of M (resp. N is isomorphic to a direct summand of M). For a cardinal number α and an R-module M, αM denotes the direct sum of α-copies of M. For a set X, we denote the cardinal number of X by $|X|$. We denote by N_0 the set of all positive integers.

DEFINITION. A ring R is *directly finite* if $xy=1$ implies $yx=1$ for all $x,y \in R$. An R-module M is *directly finite* if $\mathrm{End}_R(M)$ is directly finite. A ring R (a module M) is *directly infinite* if it is not directly finite. It is well-known that M is directly finite if and only if M is not isomorphic to a proper direct summand of M itself. A ring R is said to be a *unit-regular* ring if, for each $x \in R$, there exists a unit (i.e. an invertible element) u of R such that $xux=x$. Let s be a positive integer. Then a regular ring R is said to satisfy *s-comparability* provided that for any $x,y \in R$, either $xR \lesssim s(yR)$ or $yR \lesssim s(xR)$. Note that 1-comparability is called *the comparability axiom.*