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SYSTEMS OF NONLINEAR DIFFERENTIAL EQUATIONS RELATED TO SECOND ORDER LINEAR EQUATIONS

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0. Introduction

We will construct new nonlinear dynamical systems from linear differential equations of second order. This method is first applied by Jacobi, in 1848 ([9]), in the case of the equation

$$x(1-x)\frac{d^2y}{dx^2}+(1-2x)\frac{dy}{dx}-\frac{1}{4}y=0. \tag{0.1}$$

A solution of (0.1) is given by the complete elliptic integral

$$K=\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-xt^2)}}=\frac{\pi}{2}\theta_3^2$$

and Jacobi found a nonlinear equation:

$$(y^2y'''-15yy'y''+30y'^3)^2+32(yy''-3y'^2)^3=-\pi^2y^{10}(yy''-3y'^2)^2, \tag{0.2}$$

which is satisfied by Jacobi's elliptic theta functions θ_2, θ_3 and θ_4.

Later, Halphen rewrote (0.2) as a nonlinear dynamical system ([7]):

$$\begin{cases} X'+Y'=2XY, \\ Y'+Z'=2YZ, \\ Z'+X'=2ZX. \end{cases} \tag{0.3}$$

Halphen showed that logarithmic derivatives of theta null values satisfy (0.3). The structure of (0.3) is studied in [4] and [11]. Halphen also studied nonlinear systems deduced from generic hypergeometric equations ([8]).

In this paper, we will construct nonlinear equations from general second-order linear equations following Jacobi's idea. They are equivalent to the well-known equation written by the Schwarzian derivative. One of the aim of this paper is