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THE FEFFERMAN-PHONG INEQUALITY IN THE LOCALLY TEMPERATE WEYL CALCULUS

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1. Introduction

The Fefferman-Phong inequality, which is a two derivatives Gårding type inequality, has been proved by its authors for pseudo-differential operators with symbols in the class $S^m_{1,0}$. It has been extended by HÖRMANDER [5], in his work on the Weyl calculus, to symbols in the general class $S(m,g)$ where g is a slowly varying and temperate metric satisfying the uncertainly principle. Further works on spectral theory and singularities for nonlinear hyperbolic equations showed the necessity to relax the temperacy condition on the metric and DENCKER [3], BONY-LERNER [1] introduced new classes allowing to deal with these applications. However the Fefferman-Phong inequality was not proved for these classes. In a recent work, COLOMBINI, DEL SANTO, ZUILY [2], we have also been led to consider non temperate metrics and the above mentioned inequality was required in the proof; it turned out that these metrics were locally temperate in the sense of DENCKER [3]. The purpose of this work is then to prove the Fefferman-Phong inequality for properly supported pseudo-differential operators with symbols in locally temperate classes. Unfortunately, because of the complexity of the quantification, this inequality is still not available in the general case of the Bony-Lerner classes.

2. Notations, statement of the result, examples

We first recall some definitions taken from HÖRMANDER [5] and DENCKER [3].

Let V be an n dimensional vector space and $W=V\oplus V'$ where V' is the dual of V. Elements in V will be denoted by x and those in W by w or (x,ξ).

Let G be a metric on V, assumed to be slowly varying i.e.

$$(2.1)\quad \begin{cases} \text{there exist constants } a_0>0,\ A_0\geq 1 \text{ such that for } x,y \text{ in } V: \\ G_x(x-y)\leq a_0 \Rightarrow \dfrac{1}{A_0}\leq \dfrac{G_x}{G_y}\leq A_0 . \end{cases}$$

Let g be a metric on W which is also slowly varying i.e.