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THE AVERAGE EDGE ORDER OF TRIANGULATIONS OF 3-MANIFOLDS

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1. Introduction

Let K be a triangulation of compact 3-manifold M with $V(K)$, $E(K)$, $F(K)$ and $T(K)$ the numbers of vertices, edges, faces, and tetrahedra in K, respectively. Note that we distinguish a triangulation from a cell decomposition into a union of 3-simplices, that is, such a cell decomposition is a triangulation when the intersection of any two simplices is actually a face of each of them. The order of an edge in K is the number of triangles incident to that edge. The *average edge order* of K is then $3F(K)/E(K)$, which we will denote $\mu(K)$. Feng Luo and Richard Stong showed in [2] that for a closed 3-manifold M, $\mu(K)$ being small implies that the topology of M is fairly simple and restricts the triangulation K. This is the following theorem.

Theorem 1 [2]. *Let K be any triangulation of a closed connected 3-manifold M without boundary. Then*

(a) $3 \leq \mu(K) < 6$, *equality holds if and only if K is the triangulation of the boundary of a 4-simplex.*

(b) *For any $\varepsilon > 0$, there are triangulations K_1 and K_2 of M such that $\mu(K_1) < 4.5 + \varepsilon$ and $\mu(K_2) > 6 - \varepsilon$.*

(c) *If $\mu(K) < 4.5$, then K is a triangulation of S^3. There are an infinite number of distinct such triangulations, but for any constant $c < 4.5$ there are only finitely many triangulations K with $\mu(K) \leq c$.*

(d) *If $\mu(K) = 4.5$, then K is a triangulation of S^3, $S^2 \times S^1$, or $S^2 \tilde{\times} S^1$. Furthermore, in the last two cases, the triangulations can be described.*

The purpose of this note is to establish similar results for compact 3-manifolds with non-empty boundary. In fact we get the following theorem.

Theorem 2. *Let K be any triangulation of a compact connected 3-manifold M with non-empty boundary. Then*

(a) $2 \leq \mu(K) < 6$, *equality holds if and only if K is the triangulation of one 3-simplex.*