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ON ALMOST QF RINGS

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0. Introduction

Let R be ring. A right R-module M is said to be small, if M is small in some right R-module containing M. Clearly, any right R-module containing a non-zero injective module is not small. Harada [5] studied rings satisfying the converse of this fact and rings dual to those. In [12], Oshiro called such rings with some chain conditions right H rings and right co-H rings, respectively, and in [13], he obtained a result that a ring R is a left H ring if and only if R is a right co-H ring. On the other hand, in [7], Harada introduced notions of almost iniective modules and almost projective modules. In [8], he defined right almost QF rings and right almost QF$^{\#}$ rings by using these notions, and proved that those rings coincide with right co-H rings and right H rings, respectively. (But it seems that there is a gap in the proof of [8, Theorem 2], because indecomposable injective modules considered in [7] (so also in [8]) are only ones which are finitely generated.) Thus we have the following result:

Theorem (Harada [8], Oshiro [13]). *For a two-sided artinian ring R, the following conditions are equivalent.*

(1) *R is a right almost* QF *ring.*
(2) *R is a left almost* QF$^{\#}$ ring.
(3) *R is a right* co-H *ring.*
(4) *R is a left* H *ring.*

In this note, we give an elementary proof for the equivalence of these four rings under a slight generalization. The proof fills the gap mentioned above.

Throughout this note we always assume R is a ring with identity and J its Jacobson radical, and unless otherwise stated, "a module" means a unitary right R-module. By M_R ($_RM$) we stress what M is a right (left) R-module. Let M be a module. Then $L \leq M$ (resp. $L < M$) means L is a submodule of M (resp. $L \leq M$ and $L \neq M$). By Top(M), Soc(M) and $E(M)$, we denote the top, the socle and an injecitve hull of M, respectively. Assume every homomorphism always operates from opposite side of scalar. "Acc" ("dcc") means the ascending (descending) chain condition. When R satisfies acc on annihilator right (left) ideals, we briefly say