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GEOMETRY OF SCROLLS

OSAMU KOBAYASHI and MASAAKI UMEHARA

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1. Introduction

In this paper, we study topology and geometry of immersed curves in the plane $\boldsymbol{R}^2$ (or preferably in the sphere $\boldsymbol{R}^2 \cup \{\infty\}$).

From topological point of view, we distinguish curves by *geotopy* ([6], [2]). Two curves are said to be *geotopic* if there is a diffeomorphism between neighborhoods of the curves which takes one to the other. Geotopy preserves information on intersections. If we restrict ourselves to *normal* curves ([8]), i.e., curves whose self-intersections are transvers double points, the intersection information is represented by a *Gauss word.* A Gauss word is simply a sequence of labels of crossing points with signs. Conversely, a Gauss word determines a geotopy type of curves.

On the geometric side, we look at *vertices* of a curve. A *vertex* is a stationary point of the curvature. It is well-known that a vertex is a concept which belongs to Möbius geometry. That is, it is invariant not only under Euclidean motions, but also under inversions. We assume that curves have only finitely many vertices, none of which are located at crossings (cf. Theorem 2.5). Then a curve is divided into finitely many vertex-free curves. Since a vertex-free curve on the plane has no self-intersections (Kneser, see[5]), topological complexity of the original curve then comes from intersections of these vertex-free pieces. As a basic case, we investigate intersections of two vertex-free curves.

Figure 1.1

Figure 1.1 shows typical vertex-free curves. From their appearances, we sometimes refer to a vertex-free curve as a *scroll.* In Figure 1.1, we recognize that one is a scroll with increasing curvature and the other with decreasing curvature. Note that these monotonicity properties of curvature are independent of choice of the orientations of a curve. Thus (non-oriented) vertex-free curves fall into two classes, one with increasing curvature, the other with decreasing curvature. This rather trivial observation will be useful in the proof of our main