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ON A WAVE EQUATION CORRESPONDING TO GEODESICS

Dedicated to Professor Hideki Ozeki on his 60th birthday

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0. Introduction

For a closed curve $\gamma(x)$ in a riemannian manifold M we define its energy $E(\gamma)$ by $\|\partial_x\gamma\|^2$. The first variation $(d/dt)_{t=0}E(\gamma(t, *))$ is given by $-2\langle\partial_t\gamma, \nabla_x^2\gamma\rangle$. Therefore, its Euler-Lagrange equation is the equation of geodesics. We consider a corresponding hyperbolic equation of $\gamma=\gamma(t, x)$:

$$\text{(H)} \qquad \nabla_t^2\gamma+\mu\partial_t\gamma=\nabla_x^2\gamma,$$

where the coefficient μ represents the resistance and is usually a positive constant. This equation is locally expressed as

$$\partial_t^2\gamma^i+\Gamma_{j\,k}^i(\gamma)\partial_t\gamma^j\partial_t\gamma^k+\mu\partial_t\gamma^i=\partial_x^2\gamma^i+\Gamma_{j\,k}^i(\gamma)\partial_x\gamma^j\partial_x\gamma^k,$$

which is a semi-linear wave equation.

Eells and Sampson [1] introduced a corresponding heat equation

$$\text{(P)} \qquad \partial_t\gamma=\nabla_x^2\gamma.$$

We know that if the manifold M is compact and real analytic, then the solution of (P) exists for all time and converges to a geodesic [3].

Physically, equation (H) represents the equation of motion of a rubber band in viscous liquid. Therefore, it is likely that results similar to (P) hold. In fact we will prove the following result.

Theorem. *Let M be a complete riemannian manifold and μ a constant. Then Cauchy problem* (H) *for closed curves has a unique solution $\gamma(t, x)$ on $\boldsymbol{R}\times S^1$. If M is compact and $\mu>0$, then the solution almost converges to geodesics*; that is, $\partial_t\gamma\to0$ *and* $\nabla_x^2\gamma\to0$ *when* $t\to\infty$.

However the convergence of γ itself is still open, even on a manifold with negative sectional curvature.