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ON É. CARTAN'S SPINOR THEORY

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Introduction

This paper consists of expository remarks on the relevance of the manifold of maximal isotropic subspaces in C^{2n+1}, with respect to a nondegenerate symmetric biliear form, to the spinors of $\mathfrak{so}(2n+1, C)$, introduced in Élie Cartan's lecture notes "*Leçons sur la théorie des spineurs* I, II (1938)" ([3]), Chapitre V.

Let us denote by G^* the complex Lie group Spin $(2n+1, C)$, the universal covering group of the complex special orthogonal group $SO(2n+1, C)$, and consider the spin representation of G^*. The dimension of the representation space is 2^n. We denot by P the complex projective space of all complex lines through the origin in the representation space, and by V the G^*-orbit in P through the point determined by the highest weight vectors. Since the center of G^* leaves every point on P fixed, V is a quotient space of $SO(2n+1, C)$.

Making use of the Clifford algebra ([1]), one can study the spin representation in detail and identify V with the space of all maximal isotropic subspaces in C^{2n+1} with respect to a non-degenerate symmetric biliear form (cf. [6], Chap. IV, §9). Further, a concrete description of this projective imbedding $V \rightarrow P$ in terms of a suitable coordinate system of V can be obtained ([5], Lemma 2.1).

On the other hand, in his book, Élie Cartan introduces the above projective imbedding $V \rightarrow P$ in an explicit form without ado ([3] Chap. V, 92), and takes this setting as the starting point of his spinor theory. In this article, we attempt to shed light on this Cartan's approach.

We show first how this projective imbedding arises naturally within the context of the space V of all maxmal isotropic subspaces (§1), The process of determining coordinate transformations associated to a suitable coordinate chart covering of V leads to a holomorphic line bundle F over V with the property that, the square $F \otimes F$ is the n-th exterior product of the vector bundle whose fibre over a point V is the vector space V itself. The projective imbedding in question is determined by a vector space of holomorphic sections of the line bundle F^{-1}. In this section, a certain determinant (Lemma in **1.5**) plays a crucial role.