Ohyama, Y.
Osaka J. Math.
32 (1995), 431–450

DIFFERENTIAL RELATIONS OF THETA FUNCTIONS

YOUSUKE OHYAMA

(Received September 20, 1993)

0. Introduction

There are many applications of modular functions or modular forms in mathematics. The ring structure of modular forms is deepy studied from algebraic viewpoint. But from analytic viewpoint, there are only a few results on modular forms.

In 1881, Halphen solved a nonlinear differential system

$$\frac{d(u_1+u_2)}{dx}=u_1u_2,$$
$$\frac{d(u_2+u_3)}{dx}=u_2u_3, \tag{0.1}$$
$$\frac{d(u_3+u_1)}{dx}=u_3u_1,$$

by theta constants ([3]). He deduced the system above from the addition formula for Weierstrass' ζ-function. In 1910, Chazy considered a third-order nonlinear differential equation

$$y'''=2y'y''-3(y')^2,$$

for $y=u_1+u_2+u_3$, in his study of Painlevé type equations of third order ([2]).

Recently it is studied that a reduction of the self-dual Yang-Mills equations is related to Halphen's system ([1]). We can consider modular functions as special solutions for the self-dual Yang-Mills equations. It seems interesting to study relations between 'Integrable systems', such as the self-dual Yang-Mills equations or self-dual metrics, and modular forms.

In this paper we will study Halphen's system (0.1). Halphen showed that

(1) A solution of (0.1) is given by logarithmic derivatives of theta constants.

(2) (0.1) is $SL(2,\boldsymbol{C})$-invariant.

(3) The action of the subgroup $SL(2,\boldsymbol{Z})$ in $SL(2,\boldsymbol{C})$ induces the perrmutation