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KdV POLYNOMIALS AND Λ-OPERATOR

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1. Introduction

The purpose of the present paper is to clarify certain algebraic properties of the spectrum of the second order ordinary differential operator

$$H(u) = -\partial^2 + u(x),$$

where $u(x)$ is a meromorphic function defined in a region of the complex plane and $\partial = ' = d/dx$. The integro-differential operator

$$\Lambda(u) = \partial^{-1} \cdot \left(\frac{1}{2} u'(x) + u(x)\partial - \frac{1}{4}\partial^3 \right)$$

plays crucial role in our approach, where $A \cdot B$ denotes the product of the operators A and B. The operator $\Lambda(u)$ is usually called the *Λ-operator* or the *recursion operator*. The Λ-operator generates the infinite sequence of differential polynomials as follows; put $Z_0(u) = 1$ and define functions $Z_n(u)$, $n \in \boldsymbol{N}$ by the recurrence relation $Z_n(u) = \Lambda(u) Z_{n-1}(u)$, $n \in \boldsymbol{N}$. Then it turns out that $Z_n(u)$ are the differential polynomials in $u, u', \cdots, u^{(2n-2)}$ with constant coefficients. We call the differential polynomials $Z_n(u)$, $n \in \boldsymbol{Z}_+$ the KdV *polynomials.*

Now, let $V(u)$ be the vector space over the complex number field $\boldsymbol{C}$ spanned by $Z_n(u)$, $n \in \boldsymbol{Z}_+$, then $\Lambda(u) \in \mathrm{End}(V(u))$, i.e. $\Lambda(u)$ can be regarded as the operator in $V(u)$. If $V(u)$ is finite dimensional then the principal part of the problem concerned with $H(u)$ can be reduced to consideration of certain algebraic properties of $\Lambda(u) \in \mathrm{End}(V(u))$. We want to call this method the *Λ-algorithm.* The main purpose of the present paper is to investigate the spectrum of $H(u)$ by the Λ-algorithm.

On the other hand, the present work is deeply related to the algebraic theory of the Darboux transformation. Those problems were discussed in [18]. See also [17].

The contents of this paper are as follows. In §2, the precise definitions of the Λ-operator and the KdV polynomials are given. In §3, the expansion theorem for the KdV polynomials is obtained. In §4, the notion of Λ-rank is introduced. In §5, the spectrum $\Gamma(u)$ of the opertor $H(u)$ is defined and certain class of eigenfunctions of $H(u)$ are exactly constructed by using the Λ-operator. In §6, the problem