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BEHAVIOR OF MINIMIZING SEQUENCES FOR THE YAMABE PROBLEM

SHOICHIRO TAKAKUWA

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1. Introduction

In 1960, Yamabe [14] presented the following problem.

The Yamabe problem. *Given a compact Riemannian manifold (M,g) of dimension $n\,(\geq 3)$, find a conformally equivalent metric with constant scalar curvature.*

He reduced this problem of finding a smooth function u together with a constant λ satisfying the nonlinear eigenvalue problem

$$\text{(Y)} \qquad -\kappa\Delta_g u + Ru = \lambda u^{N-1}, \qquad u>0 \qquad \text{in } M,$$

$$\kappa = \frac{4(n-1)}{n-2}, \qquad N = \frac{2n}{n-2},$$

where Δ_g denotes the negative definite Laplacian and R is the scalar curvature of g. He attempted to solve equation (Y) by finding a positive extremal of the functional

$$Q(u) = \int_M (\kappa|\nabla u|^2 + Ru^2)dV/(\int_M |u|^N\, dV)^{2/N}. \tag{1.1}$$

He claimed that for any M quation (Y) has a solution which attains the minimum

$$\lambda(M) = \inf\{Q(u) | u \in C^\infty(M),\ u \not\equiv 0\}. \tag{1.2}$$

This constant $\lambda(M)$ is a conformal invariant, which is often called the *Yamabe invariant.* In 1968, however, Trudinger [13] discovered an error in Yamabe's proof and showed that Yamabe's proof works in case $\lambda(M)$ is bounded above by a small constant. In 1976, Aubin [1] extended Trudinger's result. He proved that if M satisfies the inequality

$$\lambda(M) < \Lambda := \lambda(S^n) = n(n-1)\,\mathrm{vol}(S^n)^{2/n}, \tag{1.3}$$