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ON THE DECOMPOSITION AND DIRECT SUMS OF MODULES

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1. Introduction. This paper studies direct sums of CS-modules. We give a number of necessary and sufficient conditions for such sums to be CS-, or quasi-continuous, modules. This question was settled in [7], in a very satisfactory way, in case the ring is commutative Noetherian. The case dealt with here is more general.

Direct sums of indecomposable modules have been investigated in great detail, in long series of papers, by M. Harada and K. Oshiro, and by B.J. Muller and S.T. Rizvi.

The well known Matlis-Papp's Theorem, for injective modules, was generalized to continuous modules in [10], and to extending modules, so to quasi-continuous modules, in [11]. The present paper generalizes such a Theorem to 1-quasi-continuous modules. As a result, we obtain that, over a right Noetherian ring, 1-quasi-continuity is equivalent to the extending property for independent family of modules.

All modules here are right-modules over a ring R. m^o denotes the annihilator in R of the element $m \in M$. $X \subseteq^e M$ and $Y \subseteq^\oplus M$ signify that X is an essential submodule, and Y is a direct summand, of M. A submodule A is closed in M if it has no proper essential extensions in M.

A module M is called a *CS-module* (*n-CS-module*), if every closed submodule A of M (A of M with U-dim$(A) \leq n$) is a direct summand. M is quasi-continuous (n-quasi-continuous) if it is CS- (n-CS) module, and satisfies the following; (C_3) ((n-C_3)): For all $X, Y \subseteq^\oplus M$ (for all $X, Y \subseteq^\oplus M$, with U-dim(X), U-dim$(Y) \leq n$), where $X \cap Y = 0$, one has $X \oplus Y \subseteq^\oplus M$. A direct sum $\oplus_{i \in I} N_i$ of submodules of M is called a local direct summand if $\oplus_{i \in F} N_i \subseteq {}^\oplus M$, for all finite subsets F of I.

For a decomposition $M = \oplus_{i \in I} M_i$, we recall the following conditions:

(A_2): For any choice of $x \in M_i$ ($i \in I$), and $m_j \in M_{i_j}$ for distinct $i_j \in I$, $j \in N$, such that $m_j^0 \supseteq x^o$, the ascending sequence $\cap_{j \geq n} m_j^o$ ($n \in N$) becomes stationary.

(A_3): For any choice of distinct $i_j \in I$ ($j \in N$) and $m_j \in M_{i_j}$, if the the sequence m_j^o is ascending, then it becomes stationary.

(lsTn) (locally semi-T-nilpotent): For every sequence f_n: $M_{i_n} \to M_{i_{n+1}}$, $n \in N$, of non-isomorphisms, with all i_n distinct, and every $x \in M_{i_0}$,