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TWO COUNTEREXAMPLES TO CORNEA'S CONJECTURE ON THIN SETS

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1. Introduction

In the paper of Cornea ([1], p. 836) is the following conjecture: A set $A \subset \boldsymbol{R}^d$ is thin at 0 if there exist v_1, v_2, $v_3 \in \boldsymbol{R}^d$ linearly independent (pairwise, if $d=2$) with $\|v_j\|=1$ and such that $T_{v_j}(A)$ is thin at $0, j=1,2,3$, where $T_v(x) := x - \langle x,v \rangle v$. We show that this conjecture fails.

We recall that the *fine topology* on $\boldsymbol{R}^d$ is the smallest topology on $\boldsymbol{R}^d$ for which all superharmonic functions are continuous in the extended sense. A set $E \subset \boldsymbol{R}^d$ is *thin* at x if x is not a fine limit point of E. The *Wiener test* relates thinness of E to the *capacity* of certain subsets of E. We note that thinness of a set at a point is related to irregularity of boundary points relative to the Dirichlet problem. For general information see [2], [3].

2. An example in $\boldsymbol{R}^2$

We denote P_x, P_y, P_z and P_w the orthogonal projections which map $\boldsymbol{R}^2$ onto a line through the origin in such a way that the points (0,1), (1,0), (1,−1) and (1,1), respectively are mapped to the origin. We set $I_2 := \{(x,y) \in \boldsymbol{R}^2,\ 0 \le x \le 1,\ 0 \le y \le 1\}$, cap denotes the logarithmic capacity.

Lemma 2.1. *Given* $\varepsilon > 0$ *there exists a set* $E \subset I_2$ *such that* $\mathrm{cap}(P_x E) < \varepsilon$, $\mathrm{cap}(P_y E) < \varepsilon$, $\mathrm{cap}(P_z E) = 0$ *and* $\mathrm{cap}(E) \ge \mathrm{cap}(P_w E) \ge \sqrt{2}/8$.

Proof. We set $A := \{(x,0) \in I_2,\ x \in \boldsymbol{Q}\}$, A is countable, hence $\mathrm{cap}(A)=0$. There exists an open set $U \supset A$ in $\boldsymbol{R}^2$ such that $\mathrm{cap}(U) < \varepsilon$. Denote $V := \{(x,0)) \in I_2\} \cap U$. We set $E := \{(x,y) \in I_2,\ (x,0) \in V,\ 0 \le y \le \varepsilon,\ x+y \in \boldsymbol{Q}\}$. Then

(i) $P_x E = V \subset U$, hence $\mathrm{cap}(P_x E) < \varepsilon$;

(ii) $P_y E = \{(0,y) \in I_2,\ 0 \le y \le \varepsilon\}$, hence $\mathrm{cap}(P_y E) < \varepsilon$;

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