Nag, S. and Sullivan, D.
Osaka J. Math.
32 (1995), 1–34

TEICHMÜLLER THEORY AND THE UNIVERSAL PERIOD MAPPING VIA QUANTUM CALCULUS AND THE $H^{1/2}$ SPACE ON THE CIRCLE

SUBHASHIS NAG and DENNIS SULLIVAN

(Received February 22, 1994)

1. Introduction

The universal Teichmüller space $T(1)$, which is a universal parameter space for all Riemann surfaces, is a complex Banach manifold that may be defined as the homogeneous space $QS(S^1)/\text{Möb}(S^1)$. Here $QS(S^1)$ denotes the group of all quasisymmetric homeomorphisms of the unit circle S^1, and $\text{Möb}(S^1)$ is the three-parameter subgroup of Möbius transformations of the unit disc (restricted to the boundary circle). There is a remarkable homogeneous Kähler complex manifold, $M = \text{Diff}(S^1)/\text{Möb}(S^1)$,—arising from applying the Kirillov-Kostant coadjoint orbit method to the C^∞-diffeomorphism group $\text{Diff}(S^1)$ of the circle ([27]). M clearly sits embedded in $T(1)$ (since any smooth diffeomorphism is quasisymmetric).

In [18] it was proved that the canonical complex-analytic and Kähler structures on these two spaces coincide under the natural injection of M into $T(1)$. (The Kähler structure on $T(1)$ is formal—the pairing converges on precisely the $H^{3/2}$ vector fields on the circle.) The relevant complex-analytic and symplectic structures on M, (and its close relative $N = \text{Diff}(S^1)/S^1$, arise from the representation theory of $\text{Diff}(S^1)$; whereas on $T(1)$ the complex structure is dictated by Teichmüller theory, and the (formal) Kähler metric is Weil-Petersson. Thus, the homogeneous space M is a complex analytic submanifold (not locally closed) in $T(1)$, carrying a canonical Kähler metric.

In subsequent work ([14], [15]) it was shown that one can canonically associate *infinite-dimensional period matrices* to the smooth points M of $T(1)$. The crucial step in this construction was a faithful representation (Segal [23]) of $\text{Diff}(S^1)$ on the Fréchet space

$$V = C^\infty \text{Maps}(S^1, \boldsymbol{R})/\boldsymbol{R}(\text{the constant maps}). \tag{1}$$

$\text{Diff}(S^1)$ acts by substitution (i.e., pullback) on the functions in V as a group of toplinear automorphisms that preserve a basic symplectic from that V carries.

In order to be able to extend the infinite dimensional period map to the full space $T(1)$, it is necessary to replace V by a suitable "completed" space that is